Dept. Of Mathematics.
Class - B.Sc. II (Maths. Hons)
Paper - IV
Name Of the Topic:
Homogeneous differential Equations

A differential equation is said to be homogeneous if it is of the form $\psi_1(x, y)\,dx = \psi_2(x, y)\,dy$ Where ψ_1 and ψ_2 are homogeneous functions of the same degree in x and y. Consequently every homogeneous eqn of the first degree and first order can be put in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Such equations can be solved by putting

$$y = vx$$

and hence $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the given equation.

Q (1) Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Soln:- We have,

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}.$$

Clearly it is a homogeneous eqn.

as x^2y as well as x^3+y^3 are homogeneous functions of the third degree.

Put

$$y = vx$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3+v^3x^3} = \frac{v}{1+v^3}$$

Or, $$x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$= \frac{v - v - v^4}{1+v^3}$$

Or, $$\frac{dx}{x} = -\frac{1+v^3}{v^4}dv$$

$$= -\left[v^{-4}dv + \frac{dv}{v}\right]$$

Integrating, we get

$$\log x + \log K = -\left[\frac{-1}{3v^3} + \log v\right]$$

$$= \frac{x^3}{3y^3} - \log\frac{y}{x}$$

where K is the constant of integration.

Or, $$\log Ky = \frac{x^3}{3y^3}$$

Or, $$Ky = e^{\frac{x^3}{3y^3}}.$$

(2) Solve $(x^2-y^2)\frac{dy}{dx}=2xy$.

Soln. we have

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$$

Put $y=vx$,

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}.$$

$$\therefore v+x\frac{dv}{dx}=\frac{2x\cdot vx}{x^2-v^2x^2}=\frac{2v}{1-v^2}$$

Or, $x\frac{dv}{dx}=\frac{2v}{1-v^2}-v=\frac{2v-v+v^2}{1-v^2}$

Or, $\frac{dx}{x}=\frac{1-v^2}{v(1+v^2)}dv=\frac{1+v^2-2v^2}{v(1+v^2)}dv$

$$=\frac{dv}{v}-\frac{2v\,dv}{1+v^2}$$

Integrating, we get

$$\log x=\log v-\log(1+v^2)+\log k$$

Where k is the const. of Intg.

$$=\log\frac{vk}{1+v^2}$$

Or, $x=\frac{vk}{1+v^2}=\frac{k\,y/x}{1+y^2/x^2}=\frac{kxy}{x^2+y^2}$

Or, $x^2+y^2=ky.$

(3) Solve $(x^2+y^2)\frac{dy}{dx} = 2xy$. (4)

Soln:- We have $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$

Put $y = vx$,

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$.

$\therefore v + x\frac{dv}{dx} = \frac{2x.vx}{x^2+v^2x^2} = \frac{2v}{1+v^2}$

Or, $x\frac{dv}{dx} = \frac{2v}{1+v^2} - v = \frac{v-v^3}{1+v^2}$

Or, $\frac{dx}{x} = \frac{1+v^2}{v(1-v^2)}dv = \left[\frac{1}{v} + \frac{1}{1-v} - \frac{1}{1+v}\right]dv$

(By partial fraction)

Integrating, we get

$\log x + \log k = \log v - \log(1-v) - \log(1+v)$,

where k is the constant of integration

Or, $\log kx = \log\frac{v}{1-v^2}$

Or, $kx = \frac{v}{1-v^2}$

Or, $kx = \frac{y/x}{1-y^2/x^2} = \frac{xy}{x^2-y^2}$

Or, $k(x^2-y^2) = y$

Note:- $\frac{1+v^2}{v(1-v^2)}dv$ can also be written

as $\frac{(1-v^2)+2v^2}{v(1-v^2)}dv$

i.e, $\frac{dv}{v} - \left(-\frac{2v\,dv}{1-v^2}\right)$. As

⑤ Solve: $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solⁿ:- Put $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} + v = v^2$

Or, $x\frac{dv}{dx} = v^2 - 2v = v(v-2)$

Or, $\frac{dx}{x} = \frac{dv}{v(v-2)} = \frac{1}{2}\left[\frac{1}{v-2} - \frac{1}{v}\right]dv$

$= \frac{1}{2}\left[\frac{dv}{v-2} - \frac{dv}{v}\right]$

Integrating, we get

$\log x = \frac{1}{2}\left[\log(v-2) - \log v\right] + \log k$

where k is the constant of integration

Or, $\log\frac{x}{k} = \frac{1}{2}\log\frac{v-2}{v} = \frac{1}{2}\log\frac{y-2x}{y}$

Or, $\log\frac{x^2}{k^2} = \log\frac{y-2x}{y}$

Or, $yx^2 = k^2(y-2x)$

Ans