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Dept. of Math.  
B.Sc. Part 2  
(Maths Honors)  
Paper 1  
(Theory of Equations)

Q. State and prove Factor Theorem

Statement: Let  $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$  be an equation of  $n^{\text{th}}$  degree. Let  $\alpha$  be a root. Then we have to prove that  $(x - \alpha)$  is a factor of  $f(x) = 0$ .

As  $\alpha$  is a root of  $f(x) = 0$ , we have  $f(\alpha) = 0$  — (1)

If  $(x - \alpha)$  is to be a factor of  $f(x) = 0$ . Then if we divide  $f(x)$  by  $(x - \alpha)$ , then there should be no remainder. The division is not exact. Let  $\phi(x)$  be the quotient and  $R$  be the remainder:

$$f(x) = \phi(x)(x - \alpha) + R$$
$$\text{Put } x = \alpha, f(x) = \phi(\alpha)(\alpha - \alpha) + R$$
$$f(x) = 0 + R \quad \text{--- (from (1))}$$

This shows that the division of  $f(x)$  by  $(x - \alpha)$  is exact. Hence  $(x - \alpha)$  is a factor.

(2)

Q. The imaginary roots of an equation with real coefficients always occur in conjugate pair.

Proof:

Let  $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$  be a general polynomial of  $n$ th degree.

Let  $\alpha + i\beta$  be one of its roots, then we have to prove that  $\alpha - i\beta$  must be another root of this equation.

As  $\alpha + i\beta$  is given to be root of the equation  $f(x) = 0$ , we have

$$f(\alpha + i\beta) = 0 \quad \text{--- (1)}$$

Also from factor theorem it is clear that  $f(x) \div \{x - (\alpha + i\beta)\}$  must be factor of the equation. If  $(\alpha - i\beta)$  be a root of the equation, then  $x - (\alpha - i\beta)$  should be another factor of the equation.

i.e.,  $(x - \alpha - i\beta)(x - \alpha + i\beta)$  should be a factor.

i.e.,  $\{ (x - \alpha)^2 - i^2 \beta^2 \}$  should be a factor.

i.e.,  $\{ (x - \alpha)^2 + \beta^2 \}$  should be a factor.

The above expression will be a factor of the given equation. If we divide  $f(x)$  by  $\{ (x - \alpha)^2 + \beta^2 \}$  then there should be no remainder.

(8)

Let us, suppose that when actual division is done  $\phi(x)$  is the quotient and  $px+q$  is the remainder.

Thus we have  $f(x) = (x-\alpha)^2 + \beta^2 \phi(x) + px+q$

Now, we put  $x = \alpha + i\beta$  then

$$f(\alpha + i\beta) = \frac{(\alpha + i\beta - \alpha)^2 + \beta^2}{\beta^2} \phi(\alpha + i\beta) + p(\alpha + i\beta) + q$$

$$\text{i.e., } 0 = \frac{i^2\beta^2 + \beta^2}{\beta^2} \phi(\alpha + i\beta) + p(\alpha + i\beta) + q$$

$$\text{i.e., } 0 = (-\beta^2 + \beta^2) \phi(\alpha + i\beta) + p(\alpha + i\beta) + q$$

$$\text{i.e., } p(\alpha + i\beta) + q = 0.$$

$$p\alpha + q + ip\beta = 0.$$

But we know that if a complex number is equal to zero. Then its real and imaginary part are separately equal to zero.

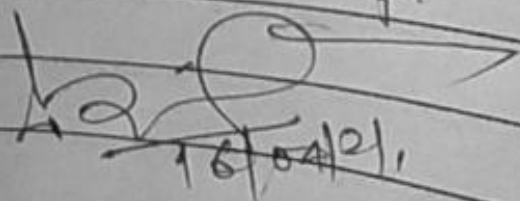
$$p\alpha + q = 0, \quad p\beta = 0.$$

$$\text{i.e., } p = 0 \text{ as } \beta \neq 0$$

$$\therefore 0\alpha + q = 0 \Rightarrow q = 0.$$

As  $p$  and  $q$  are separately equal to zero. It is obvious that the remainder  $px+q = 0$ . Thus we observe that the division of  $f(x)$  is exactly by the product of factors.

Hence  $\alpha + i\beta$  is another root.

  
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