

① Date: 16/04/21

Dept. of Mathematics

B.Sc. Part II

(Maths. Honors)

Paper - III

Real Analysis: (Lebesgue theory)

Theorem: - if α and β are two cuts then
prove that $\alpha\beta = \beta\alpha$.

Proof: - Let $\alpha = (A_1, A_2)$ and $\beta = (B_1, B_2)$
be any two cuts. then to prove that
 $(A_1, A_2), (B_1, B_2) = (B_1, B_2), (A_1, A_2)$
i.e., $\alpha\beta = \beta\alpha$.

Let $(U_1, U_2) = (A_1, A_2), (B_1, B_2)$

and $(V_1, V_2) = (B_1, B_2), (A_1, A_2)$

Let $u_1 \in U_1$ and $v_1 \in V_1$

Then u_1 is of the form $x_1 y_1$ where
 $x_1 \in A_1$ and $y_1 \in B_1$, and v_1 is of the
form $y_1 x_1$, where $x_1 \in A_1, y_1 \in B_1$. Since
the commutative law holds for ra-
tional numbers. so we have

$$x_1 y_1 = y_1 x_1$$

which shows that $U_1 = V_1$ and conse-
quently $(U_1, U_2) = (V_1, V_2)$

Thus $(A_1, A_2), (B_1, B_2) = (B_1, B_2), (A_1, A_2)$

②
Theorem! - State and prove
Dedekind's Theorem.

Statement! Let R_1 and $R_2 \subset R$ where
 R is the set of real number
Such that

(i) $R_1 \neq \emptyset, R_2 \neq \emptyset$ i.e. each class exists

(ii) $R_1 \cup R_2 = R$

(iii) if $x \in R_1, y \in R_1 \Rightarrow x < y$

i.e. every member of R_1 is less than every
member of R_2 .

Then \exists a greatest number of R_1 or a
least number of R_2 .

Proof! - Here we shall show that only two
types of section are possible.

(i) either R_1 has a greatest member
or

(ii) R_2 has a least member.

Let A_1 and A_2 are the set
of rational (real) of R_1 and R_2 res-
pectively. Consider the section (A_1, A_2)
with $A_1 \subset R_1$ and $A_2 \subset R_2$.

if x and y are rational numbers, then
 $x \in A_1 \Rightarrow x \in R_1$ and $y \in A_2$ and con-
versely since x and y are rational
numbers.

We find that $x \in R_1 \rightarrow x \in A_1$ (3)

and $y \in R_2 \rightarrow y \in A_2$

Here following three cases arise:

CASE I:- The lower class A_1 has greatest member (say l) and the upper class A_2 has no least member.

Since A_1 is a proper subset of A_2 . So $l \in A_1 \rightarrow l \in R_1$. Let if possible l is not the greatest member of A_1 but l' is the greatest member of A_1 . l' may be a rational. Then $l' > l$.

But R is a dense set and so there exist an infinite number of rationals between l and l' . These rational numbers are less than l' . So, they will belong to A_1 . Hence there exist an infinite number of rationals in A_1 which are greater than l . This l is not the greatest member of A_1 . This contradiction shows that l is the greatest member of R_1 .

$\therefore l$ is the greatest member of $A_1 \rightarrow l$ is the greatest member of R_1 .

CASE II:- A_1 has no greatest member and A_2 has a least member (say m). Since as above, we can say that R_1 has no greatest member and R_2 has a least member m .

CASE III -

(4)

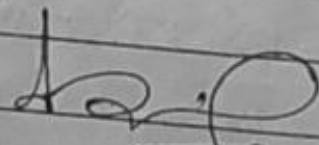
A_1 has no greatest member and A_2 has no least member.

From above discussion, we find that there are only two types of sections of real numbers!

(i) R_1 has a greatest member but R_2 has no least.

(ii) R_1 has no greatest member but R_2 has a least.

Hence in either case R_1 has a greatest member or R_2 has a least. Third possibility, namely R_1 has no greatest member and R_2 has no least member does not find a place in the section of real numbers. This completes the result.



12/04/21

Dr. S. Suresh Babu
Dept. of Maths
D.K. College,
Dumkaon