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Dept. Of Mathematics.

B.Sc. Part II (Maths. Hons)

Paper- III

Real Analysis

Theorem:- State and explain Dedkind's
(a) Order - Completeness theorem for real numbers.
(b) Order incompleteness theorem for irrational number.

A (a) Statement (i) There does not exist a section (R_1, R_2) of the set of real numbers in which R_1 has no greatest member and R_2 has no least member.
(ii) There are no gaps in the system of real numbers.
(iii) The set of real numbers is complete i.e the continuum is closed.

(b) Statement:- (i) There exists a section (L, R) of the set of rational numbers in which L has no greatest member and R has no least member
(ii) There are gaps is the system of rational numbers.
(iii) The set of rational numbers is Incomplete.

Explain!

We know that in the system of rational numbers, there is a gap which is represented by the section in which L has no greatest member and R has no least member. By introducing of a new number as a real number more general than that of a rational number, that gap is filled. But the section of (R_1, R_2) of real numbers in which R_1 has no greatest member and R_2 has no least member does not occur and thus the section of real numbers in to two classes R_1 and R_2 does not produce a new number different from a real number i.e, there is no gap in the system of real numbers.

Conclusion: (i) When a system of numbers has no gaps then it is said to be complete.

(ii) When a system of numbers has gaps then it is said to be incomplete.

Hence the system of real numbers is complete while the system of rational numbers is incomplete.

Theorem :- State and prove Archimedean property of Rational numbers.

Statement: If x and y are two positive rational numbers such that $x > b$, $x \neq 0$. then there exist a positive integer n such that $ny > x$,

[P.T.O]

Proof:- Let us contrary that

$ny < x \ \forall\ n \in \mathbb{Z}$.

then $ny < x \Rightarrow n\frac{y}{x} < 1$.

$\Rightarrow n\frac{y}{x} < m \ \forall\ m \in \mathbb{Z}$.

$\Rightarrow n\frac{y}{m} < x$

$\Rightarrow \frac{n}{m} < \frac{x}{y}$ ——— (i)

But $\frac{n}{m}$ is a +ve real rational number say a. Here $a < \frac{x}{y} \ \forall\ a \in \mathbb{Q}^+$ [i.e set of +ve rational numbers] But this is a false. So this lead contradiction to our supposition that $ny < x \ \forall\ n \in \mathbb{Z}$.

This completes the theorem.

Thm:- Show that there are infinity of rational numbers between any two different rational numbers.

Proof:- Let x and y are any two different rational numbers and let $x < y$

$\Rightarrow xk < yk$ [where $k > 0$ rational numbers] (i)

$\Rightarrow x + xk < x + yk$ [by adding x both sides] (ii)

$\Rightarrow x(1+k) < x + yk$

$\Rightarrow x < \frac{x+yk}{1+k}$ ——— (iii)

Also adding ~~yx~~ y to both sides of (i), we get

$x + yk < y + yk$

$x + yk < y(1+k)$

$\Rightarrow \frac{x+yk}{1+k} < y$ (iv)

By (iii) & (iv), we get

$x < \frac{x+yk}{1+k} < y$ $\therefore$

Here we choose the value of k as infinity so $\frac{x+yk}{1+k}$ gives infinite value. So there are infinite rational nos. between any two different rational nos.