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Dept. of Mathematics.
B. Sc. Part I (Maths. Hons)
Paper - II
Integral Calculus

Name of the Topic: Integration of rational function (by Partial fractions)

A function of the form $\frac{F(x)}{\psi(x)}$ is called a rational algebraic function if $F(x)$ and $\psi(x)$ are polynomial in x. Thus

$$\frac{x}{a^2+x^2}, \quad \frac{2x^2-3x+1}{x^2+x+1}, \quad \frac{1}{a^2-x^2}, \quad \frac{5x^3+7}{4x^2+3x+1} \text{ etc.}$$

are all Rational functions.

Additional Standard forms for Integration of Rational functions:

The following Integrals are very important for integration of rational functions.

(1) $\int \frac{1}{x^2+a^2}\,dx$.

Put $x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$

Now, $I = \int \frac{a\sec^2\theta\,d\theta}{a^2\tan^2\theta + a^2}$

(2)

$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\cdot\theta$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a}$$

(ii) $\int \frac{1}{x^2-a^2}\,dx, \ (x>a)$

Here, $I = \int \frac{1}{x^2-a^2}\,dx$

$$= \int \frac{1}{(x-a)(x+a)}\,dx$$

$$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx.$$

$$= \frac{1}{2a}\int \left[\frac{(x+a)}{(x-a)(x+a)} - \frac{(x-a)}{(x-a)(x+a)}\right]dx$$

$$= \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{x-a}\,dx - \int \frac{1}{x+a}\,dx\right]$$

$$= \frac{1}{2a}\left[\log(x-a) - \log(x+a)\right]$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a}, \ (x>a)$$

(iii) $\int \frac{1}{a^2-x^2}\,dx \quad (x<a)$ (5)

Here, $I = \int \frac{1}{a^2-x^2}\,dx$

$$= \int \frac{1}{(a-x)(a+x)}\,dx$$

$$= \frac{1}{2a}\int \frac{(a-x)+(a+x)}{(a-x)(a+x)}\,dx$$

$$= \frac{1}{2a}\int \left[\frac{a-x}{(a-x)(a+x)} + \frac{a+x}{(a-x)(a+x)}\right] dx$$

$$= \frac{1}{2a}\int \left[\frac{1}{a+x} + \frac{1}{a-x}\right] dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{a+x}\,dx + \int \frac{1}{a-x}\,dx\right]$$

$$= \frac{1}{2a}\left[\log(a+x) - \log(a-x)\right]$$

$$= \frac{1}{2a}\log\frac{a+x}{a-x}, \quad (x<a)$$