Date: 27.04.21 (1)

Dept. of Mathematics.
B.Sc. Part II (Maths Hons.)
Paper - III

Name of the Topic:
Abstract Algebra (Groups)

Thm:- To State and prove Uniqueness of identity.

Or,

The identity element in a group is unique.

Proof:- An element $e \in G$ is said to be an identity for the operation in G if

$$ae = ea = a \ \forall \ a \in G$$

We want to prove that e is unique. If possible, let e and e' be two identity elements both belong to G.

Then e is an identity element

$\Rightarrow ee' = e'$

Again e' is an identity element

$\Rightarrow ee' = e.$

So we find that

$$e' = e.$$

Hence e is unique.

Thm:- To State and prove Uniqueness of inverse.

Or,

The inverse of an element in a group is unique.

Proof:- An element b is said to be inverse of an element a if

$$ab = ba = e \in G \text{ for } a, b \in G$$

Let the inverse be not unique.

let b and $c \in G$ be two distinct inverse of a, then we have

$$ab = ba = e \text{ and } ac = ca = e.$$

Now, $(ba)c = ce = b$, $a, b, c \in G$

But $(ba)c = b(ac)$, by Associative Property

$\therefore c = b.$

Hence the inverse is unique.

Thm:- (Solvability of equations in a Group)

The equation $ax = b$ and $ya = b$. $a, b, x, y \in G$ have unique solution in G.

Proof:- $\because ax = b$

Substituting $a^{-1}b$ for x

$\therefore ax = b$, we get

$a(a^{-1}b) = b.$

i.e, $(aa^{-1})b = b$ (by associative property) i.e, $eb = b$ (by existence of inverse)

i.e, $b = b$ (by existence of identity)

Thus the equation $ax = b$ is satisfied by $x = a^{-1}b$.

Also $a^{-1}b \in G$. Since $a^{-1} \in G$ and $b \in G$

furthermore, the solution is unique.

To prove this,

let if possible x_1 and x_2 be two solution of $ax = b$. then

$$ax_1 = b \text{ and } ax_2 = b.$$

$$\therefore ax_1 = ax_2$$

$$\Rightarrow x_1 = x_2 \text{ (by Cancellation Law)}$$

Hence $a^{-1}b$ is unique solution of the eqn $ax = b$

Similarly, we can show that the equation $ya = b$ has a unique solution ba^{-1}.

<u>Defining a Group</u> (<u>IInd another way</u>)

:

A non-empty set G with a binary operation is a group if and only if

(i) The binary operation is associative and

(ii) The equation $ax = b$ and $ya = b$; $a, b \in G$ have unique solution in G.

The equivalence of different definitions of group can be easily established

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Thm:- If $a \in G$ then $(a^{-1})^{-1} = a$ i.e, the inverse of the inverse of an element of a group is the element itself.

Proof:- The solution of the equation $a^{-1}x = e$ is $x = ae = a$

But the solution of $a^{-1}x = e$ is the inverse of a^{-1}, by the last theorem.

$$\therefore (a^{-1})^{-1} = a$$

Hence the inverse of a^{-1} is a.

Thm:- The Inverse Of the product of two elements of a group is product of the elements in the reverse order.

i.e $(ab)^{-1} = b^{-1}a^{-1}$, $a, b \in G$.

Proof:- we have

$$\begin{aligned}(ab)(b^{-1}a^{-1}) &= [(ab)b^{-1}]a^{-1} \text{ (by associative axiom)}\\ &= [a(bb^{-1})]a^{-1} \text{ (by associative axiom)}\\ &= (ae)a^{-1} \text{ (by existence of identity)}\\ &= e \text{ (by existence of inverse)}\end{aligned}$$

Again

$$\begin{aligned}(b^{-1}a^{-1})(ab) &= b^{-1}[a^{-1}(ab)] \text{ (by Associative axiom)}\\ &= b^{-1}[(a^{-1}a)b] \text{ (by Associative axiom)}\\ &= b^{-1}(eb) \text{ (by existence of inverse)}\\ &= b^{-1}b \text{ (by existence of identity)}\\ &= e \text{ (by existence of inverse)}\end{aligned}$$

Hence, by the definition of inverse

$$(ab)^{-1} = b^{-1}a^{-1}.$$

27.04.21