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Dept. of Mathematics.

B.Sc. Part-I (Maths. Hons).

Paper - II

Name of the Topic: Integration by Partial Fraction.

TYPE 3:

When the denominator contains quadratic factors such as $a_1x^2+b_1x+c_1$, $a_2x^2+b_2x+c_2$ etc. then the given fraction can be written as

$$\frac{Ax+B}{a_1x^2+b_1x+c_1}+\frac{Cx+D}{a_2x^2+b_2x+c_2}+\cdots$$

Where A, B, C, D, --- are constants to be determined and then the integration can be performed. The following example elucidates the process.

Ex. Integrate $\int\frac{dx}{x^3-1}$

Solⁿ:- Let $\frac{1}{x^3-1}$ i.e. $\frac{1}{(x-1)(x^2+x+1)}=$

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

Or, $1=A(x^2+x+1)+(Bx+C)(x-1)$

(5)

Equating the coefficients of x^2, x, x^0 successively, we get

$$A + B = 0 \quad \text{—— (1)}$$

$$A - B + C = 0 \quad \text{—— (2)}$$

$$A - C = 1 \quad \text{—— (3)}$$

From (1) and (3), we have $A = -B, A = C + 1$

$\therefore$ From (2), $A + A + A - 1 = 0$, or $A = \frac{1}{3}$.

$\therefore B = -\frac{1}{3}, \quad C = -\frac{2}{3}$.

$$\therefore I = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x+1+3}{x^2+x+1}\,dx.$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\sqrt{3}/2} + k$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + K$$

where K is the constant of Integration.

TYPE 4:

When the denominator contains repeated quadratic factors such as $(a_1x^2 + b_1x + c_1)^3$, etc, then the given fraction can be expressed as

$$\frac{Ax+B}{a_1x^2+b_1x+c_1}+\frac{Cx+D}{(a_1x^2+b_1x+c_1)^2}+\frac{Ex+F}{(a_1x^2+b_1x+c_1)^3}+\cdots$$

Where A, B, C, D, E, F, --- are constants to be evaluated and finally the integration can be carried out.

Some integral functions can be reduced to algebraic rational functions after proper substitutions. the following example illustrates the process.

Integrate $\int \frac{dx}{\sin x+\sin 2x}$

Here, $I=\int \frac{dx}{\sin x+2\sin x\cos x}$

$$=\int \frac{dx}{\sin x(1+2\cos x)}$$

$$=\int \frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

Put $\cos x=t$, $\therefore -\sin x\,dx=dt$

$$\therefore I=-\int \frac{dt}{(1-t^2)(1+2t)}$$

Now, $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$\therefore 1 = A(1+t)(1+2t) + B(1+2t)(1-t) + C(1-t)(1+t)$

Putting $t = 1, -1, \frac{-1}{2}$ successively, we get

$A = \frac{1}{6}, B = \frac{-1}{2}, C = \frac{4}{3}.$

$\therefore I = \frac{-1}{6}\int \frac{dt}{1-t} + \frac{1}{2}\int \frac{dt}{1+t} - \frac{4}{3}\int \frac{dt}{1+2t}$

$= \frac{1}{6}\log(1-t) + \frac{1}{2}\log(1+t) - \frac{2}{3}\log(1+2t) + k.$

$= \frac{1}{6}\log(1-\cos 5x) + \frac{1}{2}\log(1+\cos 5x) - \frac{2}{3}\log(1+2\cos 5x) + k.$

where k is the constants of integration