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Mathematics.

B.Sc. Part-I (M.H)

Paper-I

Topic: Gregory's Series (H. Trigo.)

Theorem

To prove that if $-1 \leq x \leq +1$.

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \text{ to } \infty$$

Sol:- According to the Art., we have

$$\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}.$$

Putting $\alpha = 1$ and $\beta = x$ in this, we have

$$\log(1+ix) = \frac{1}{2}\log(1+x^2) + i\tan^{-1}x \quad \text{—(1)}$$

(Only taking the principle value)

Also, when $|x| \leq 1$

$$\log(1+ix) = ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} - \frac{(ix)^4}{4} + \cdots$$

$$ix + \frac{x^2}{2} - \frac{ix^3}{3} - \frac{x^4}{4} + \frac{ix^5}{5} \cdots$$

$$= \left(\frac{x^2}{2} - \frac{x^4}{4} + \cdots\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) \quad \text{--- (2)}$$

Now equating the imaginary parts from (1) and (2), we get

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

when $|x| \leq 1$ --- (3)

Putting $x = \tan\theta$ so that $\theta = \tan^{-1}x$. The above series can be written as

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \text{ to } \infty \quad \text{--- (4)}$$

Here $|\theta| \leq \frac{\pi}{4}$ i.e. $-\frac{\pi}{4} < \theta \leq +\frac{\pi}{4}$

Convergency of the Gregory's Series

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \text{ ad. inf.}$$

Here,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^n \tan^{2n+1}\theta}{2n+1} \cdot \frac{2n-1}{(-1)^{n-1}\tan^{2n-1}\theta}\right|$$

$$= \lim_{n\to\infty} \frac{2n-1}{2n+1} \cdot \tan^2\theta$$

$$= \tan^2\theta.$$

Hence,

the Series is Convergent for all values of θ for which

$$|\tan^2\theta| < 1$$

i.e, $-1 < \tan\theta < +1$

i.e, $n\pi - \frac{\pi}{4} < \theta < n\pi + \frac{\pi}{4}$

for $\tan\theta = \pm 1$, also, the series on the R.H.S is a convergent series, for by putting $\tan\theta = \pm 1$ the series on the R.H.S is an alternating series which is convergent.

Hence the Gregory's series is convergent provided $|\tan\theta| \leq 1$.

i.e, $n\pi - \pi/4 \leq \theta \leq n\pi + \pi/4$.

Note:- While discussing convergency of the Gregory-Series we had pointed out that the Gregory Series is convergent provided θ lies between the interval of the form $n\pi - \pi/4$ to $n\pi + \pi/4$. Hence it follows as a consequences, that the series would not be convergent in an interval whose extreme ends are not of the form $n\pi \pm \pi/4$. for example, the Gregory Series is not convergent if θ lies between, say

$$n\pi + \frac{\pi}{4} \text{ and } n\pi + \frac{3\pi}{4}.$$

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