

04.06.2021

Date _____
Page _____

MATHEMATICS

B. Sc. Part-II (Maths Honours)

Paper - IV

Topic: Orthogonal trajectory
(diff. eqn.)

Find the orthogonal trajectory of the family of circles $x^2 + y^2 = 2ax$ each of which touches the y-axis at the origin.

Solⁿ: Here, given $x^2 + y^2 = 2ax$ — (1)
Differentiating (1) w.r.t. x , we get
 $2x + 2y \frac{dy}{dx} = 2a$

$$\Rightarrow x + y \frac{dy}{dx} = a \quad \text{--- (2)}$$

Eliminating (a) between (1) and (2), we get,

$$x^2 + y^2 = 2x \left(x + y \frac{dy}{dx} \right)$$

$$\Rightarrow x^2 + y^2 = 2x^2 + 2xy \frac{dy}{dx}$$

$$\Rightarrow x^2 + 2xy \frac{dy}{dx} - y^2 = 0 \quad \text{--- (3)}$$

The differential equation of the orthogonal trajectory is obtained by replacing $\frac{dy}{dx}$

Date _____
Page _____

by $\frac{1}{\frac{dy}{dx}}$ and hence we get

$$x^2 + 2xy \left(\frac{1}{\frac{dy}{dx}} \right) - y^2 = 0$$

$$\Rightarrow x^2 - 2xy \frac{dx}{dy} - y^2 = 0.$$

$$\Rightarrow y^2 - x^2 + 2xy \frac{dx}{dy} = 0 \quad \text{--- (4)}$$

We find that (3) and (4) are the same except that x and y are interchanged.

Hence the integral of (4) are the same except that x and y are interchanged and must be $y^2 + x^2 = 2by$ i.e. $x^2 + y^2 = 2by$ which is another set of circles, each of which touches the x -axis at the origin.

~~Q.3~~ Find the orthogonal trajectory of $r^n \sin n\theta = a^n$.

Solⁿ Given $r^n \sin n\theta = a^n$
taking logarithm on both sides.

we get,

$$\log r^n + \log \sin n\theta = n \log a.$$

$$\Rightarrow n \log r + \log \sin n\theta = n \log a$$

Now, differentiating w.r.t θ ,

we get

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{\cos n\theta}{\sin n\theta} \cdot n = 0.$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} + \cot n\theta = 0$$

$$\Rightarrow r \frac{d\theta}{dr} = -\tan n\theta.$$

Therefore the differential equation of the orthogonal trajectory which is obtained by replacing $r \frac{d\theta}{dr}$ by $-\frac{1}{r} \cdot \frac{dr}{d\theta}$ is

$$-\frac{1}{r} \cdot \frac{dr}{d\theta} = -\tan n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \tan n\theta.$$

$$\Rightarrow \frac{dr}{r} = \tan n\theta \cdot d\theta$$

$$\Rightarrow \int \frac{dr}{r} = \int \tan n\theta \cdot d\theta$$

$$\Rightarrow \log r = \frac{1}{n} \log \sec n\theta + K.$$

$$\Rightarrow \log r = \frac{1}{n} \log \sec n\theta + \log C$$

Date _____
Page _____

$$\Rightarrow \log \frac{r}{c} = \frac{1}{n} \log \sec \theta$$

$$\Rightarrow n \log \frac{r}{c} = \log \sec \theta$$

$$\Rightarrow \log \left(\frac{r}{c} \right)^n = \log \sec \theta$$

$$\Rightarrow \left(\frac{r}{c} \right)^n = \sec \theta$$

$$\Rightarrow r^n = c^n \sec \theta$$

$$\Rightarrow r^n \cos \theta = c^n$$

Which is the required equation of the trajectory.

Imp Question! Prove that the system of confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \text{ is self-orthogonal.}$$

Sol! Differentiating the given equation, we get

$$\frac{2x}{a^2 + \lambda} + \frac{2y}{b^2 + \lambda} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2x}{a^2 + \lambda} + \frac{2yp}{b^2 + \lambda} = 0 \text{ where } p = \frac{dy}{dx}.$$

$$\Rightarrow x(b^2 + \lambda) + yp(a^2 + \lambda) = 0$$

$$\Rightarrow (b^2x + a^2yp) + \lambda(x + yp) = 0$$

$$\Rightarrow \lambda = - \frac{b^2x + a^2yp}{x + yp}$$

$$\therefore a^2 + \lambda = a^2 - \frac{b^2x + a^2yp}{x + yp}$$

$$= \frac{(a^2 - b^2)x}{x + yp}$$

$$\text{and } b^2 + \lambda = b^2 - \frac{b^2x + a^2yp}{x + yp}$$

$$= - \frac{(a^2 - b^2)yp}{x + yp}$$

Putting the values of $a^2 + \lambda$ and $b^2 + \lambda$ in the given eqn, we get

$$\frac{x^2(x + yp)}{(a^2 - b^2)x} - \frac{y^2(x + yp)}{(a^2 - b^2)yp} = 1$$

$$\Rightarrow \frac{x + yp}{a^2 - b^2} \left(x - \frac{y}{p} \right) = 1$$

$$\Rightarrow (x + yp) \left(x - \frac{y}{p} \right) = a^2 - b^2 \quad \text{--- (1)}$$

In order to find its orthogonal trajectory, we have to replace p by $-\frac{1}{p}$ and we get,

$$p \left(x - \frac{y}{p} \right) (x + yp) = a^2 - b^2 \quad \text{--- (2)}$$

Now (1) and (2) are the same and hence the system of conics is self-orthogonal.

Dr. Birendra K. Asst Prof. Dept. of Math. D.K. College, Dumraon