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Mathematics.

B. Sc. Part-I (M.H)

Paper-I

Topic: Gregory's Series (Higher Trigonometry)

Examples of Particular Cases.

(i) if θ lies between $\pi - \frac{\pi}{4}$ and $\pi + \frac{\pi}{4}$ i.e. if θ lies between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ then we have, $n=1$ and from the eqn $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta$, we get

$$\theta - \pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \text{ to } \infty$$

(ii) if $\pi - \frac{\pi}{4} \leq \theta \leq -\pi + \frac{\pi}{4}$ i.e. if $-\frac{5\pi}{4} \leq \theta \leq -\frac{3}{4}\pi$, then we have $n=-1$ and we get from equations

$$\theta + \pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots \text{ to } \infty$$

(i) Evaluation of π

One of the chief uses of Gregory's Series is to find the value of π. Thus if we put $x=1$ in (3), then since $\tan^{-1} 1 = \frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

But the difficulty in using this series lies in the fact that we have to take a large number of terms in finding the value of π correct to a given number of decimal places. Therefore we take recourse to another series; e.g.

(1) Euler's Series: it can be proved easily that

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

which when expanded by Gregory Series, becomes

$$= \left[\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \frac{1}{7}\cdot\frac{1}{2^7} + \cdots\right]$$

$$+\left[\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\frac{1}{7}\cdot\frac{1}{3^7}+\cdots\right]$$

etc.

(ii) Machin's Series:

$$\because 2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}$$

$$=\tan^{-1}\frac{10}{24}=\tan^{-1}\frac{5}{12}$$

$$\therefore 4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{5}{12}$$

$$=\tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}}=\tan^{-1}\frac{120}{119}.$$

$$\therefore 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}$$

$$=\tan^{-1}1=\frac{\pi}{4}.$$

$$\therefore \frac{\pi}{4}=4\left[\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-\frac{1}{7\cdot 5^7}+\cdots\right]$$

$$-\left[\frac{1}{239}-\frac{1}{3\cdot 239^3}+\frac{1}{5\cdot 239^5}-\frac{1}{7\cdot 239^7}+\cdots\right]$$

Now

$$\frac{1}{5} = .2$$

$$\frac{1}{3} \cdot \frac{1}{5^3} = \frac{1}{375} = 0.0026666$$

$$\frac{1}{5 \cdot 5^5} = \frac{1}{15625} = 0.000064$$

Also $\frac{1}{239} = 0.0041841$

Hence $\frac{\pi}{4} = 4\left[0.2 - 0.0026666 + 0.000064 - \cdots\right]$

$- 0.0041841$

$= 0.7853983.$

$\therefore \pi = 4 \times 0.7853983$

$= 3.14159.$

This is the correct value of π to 5 decimal places.

(iii) Rutherford's Series:

it can be shown that

$$\tan^{-1}\frac{1}{239} = \tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}.$$

Thus $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} -$

$\tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.

Both the series of $\tan^{-1}\frac{1}{70}$ and $\tan^{-1}\frac{1}{99}$ are convenient for numerical calculation.

Q: Write down the value of n when θ lies between $\frac{19\pi}{4}$ and $\frac{21\pi}{4}$ for the truth of general theorem.

$$\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

Soln: We know that the general theorem, $\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$

is valid if θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$. It is given here that θ lies between $\frac{19\pi}{4}$ and $\frac{21\pi}{4}$ i.e $5\pi - \frac{\pi}{4}$ and $5\pi + \frac{\pi}{4}$

$\therefore n = 5$.

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