

08/16/2021.

Date _____
Page _____

Mathematics - Home.

B.Sc. Part-I

Paper-I

Name of the Topic:

Gregory's Series
(Higher Trigonometry)

Ex-1. Prove that

$$\frac{\tan^{-1} \cos \theta + \sin \theta}{\cos \theta - \sin \theta}$$

$$= n\pi + \frac{\pi}{4} + \tan \theta - \frac{1}{3} \tan^3 \theta$$

$$+ \frac{1}{5} \tan^5 \theta - \dots$$

Soln:

$$\text{L.H.S.} = \frac{\tan^{-1} \cos \theta + \sin \theta}{\cos \theta - \sin \theta}$$

$$= \frac{\tan^{-1} 1 + \tan \theta}{1 - \tan \theta}$$

dividing the num. and den. by $\cos \theta$.

$$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\}$$

$$= n\pi + \frac{\pi}{4} + \theta$$

$$= n\pi + \frac{\pi}{4} + \tan \theta - \frac{1}{3} \tan^3 \theta$$

$$+ \frac{1}{5} \tan^5 \theta \text{ (by Gregory's series)}$$

Q. If $0 \leq x \leq 1$, Show that

$$\tan^{-1} \frac{1+x}{1-x} = \frac{\pi}{4} + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Solⁿ: Let $x = \tan \theta$.

Then since $0 \leq x \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$

$$\text{Then, L.H.S} = \tan^{-1} \frac{1+x}{1-x}$$

$$= \tan^{-1} \frac{1+\tan \theta}{1-\tan \theta}$$

$$= \tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \theta \right) \right\}$$

$$= \frac{\pi}{4} + \theta$$

$$= \frac{\pi}{4} + \tan^{-1} x$$

$$= \frac{\pi}{4} + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

(by Gregory's series)

Ex: If $-(\sqrt{2}-1) < x < \sqrt{2}-1$, Prove

that $2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right)$

$$= \frac{2x}{1-x^2} - \frac{1}{3} \left(\frac{2x}{1-x^2} \right)^3 +$$

$$\frac{1}{5} \left(\frac{2x}{1-x^2} \right)^5 - \dots$$

Solution

Date _____

Page _____

$$\text{L.H.S} = 2 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right)$$

$$= 2 \tan^{-1} x \quad (\text{to } \infty)$$

$$= \tan^{-1} \frac{2x}{1-x^2}$$

Now, $\tan^{-1} \frac{2x}{1-x^2}$ can be

expanded by Gregory's Series if $\frac{2x}{1-x^2} < 1$

i.e. if $\tan^{-1} \frac{2x}{1-x^2} < 45^\circ$

i.e. if $2 \tan^{-1} x < 45^\circ$

i.e. if $\tan^{-1} x < 22\frac{1}{2}^\circ$

But we know that

$$\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$$

Hence if $|x| < \sqrt{2} - 1$ (which is given here),

then the inverse function $\tan^{-1} \frac{2x}{1-x^2}$ can be expanded

by Gregory's Series.

Thus

$$\tan^{-1} \frac{2x}{1-x^2} = \frac{2x}{1-x^2} - \frac{1}{3} \left(\frac{2x}{1-x^2} \right)^3 +$$

$$\frac{1}{5} \left(\frac{2x}{1-x^2} \right)^5 - \dots = \text{R.H.S.}$$

Ex. 1. If $0 < \theta < \frac{\pi}{4}$, prove that

$$\log(\sec \theta) = \frac{1}{2} \tan^2 \theta - \frac{1}{4} \tan^4 \theta + \frac{1}{6} \tan^6 \theta - \dots$$

Sol.:- Putting $\tan \theta = x$ in the R.H.S, we get

$$\text{R.H.S} = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \dots$$

$$\text{But } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots, \quad |x| < 1.$$

Now replacing x by (ix) , we get

$$\log(1+ix) = ix - \frac{i^2 x^2}{2} + \frac{i^3 x^3}{3} - \frac{i^4 x^4}{4} + \frac{i^5 x^5}{5} - \frac{i^6 x^6}{6} + \dots$$

$$= ix + \frac{x^2}{2} - i \frac{x^3}{3} - \frac{x^4}{4} + i \frac{x^5}{5} + \frac{x^6}{6} - \dots$$

$$= \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \dots \right) +$$

$$i \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right) \quad \text{--- (1)}$$

Date _____
Page _____

$$\text{Now, } \log(1+ix)$$

$$= \frac{1}{2} \log(1^2+x^2) + i \tan^{-1}x$$

$$= \frac{1}{2} \log(1+x^2) + i \tan^{-1}x \quad (2)$$

Here equating the real part from (1) and (2), we get

$$\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \dots$$

$$= \frac{1}{2} \log(1+x^2)$$

$$\text{i.e., } \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \dots$$

$$= \frac{1}{2} \log(1+\tan^2\theta),$$

provided $0 < \tan\theta < 1$

$$\text{i.e., } 0 < \theta < \frac{\pi}{4}$$

$$= \frac{1}{2} \log(\sec^2\theta) = \frac{1}{2} \cdot 2 \log(\sec\theta)$$

$$= \log(\sec\theta) = \text{L.H.S.}$$

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