09.08.2021.

Mathematics (Hons)
B. Sc. Part-II
Paper - IV

Topic: Equation with Right hand member $x=0$:

<u>Auxiliary Equation</u>

Let the differential equation with $x=0$ be

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots + P_n y = 0 \quad \text{--- (1)}$$

Suppose, $y = e^{mx}$ is a trial solution of (1)

if $y = e^{mx}$ is a solution of (1), then it must satisfy the equation (1).

Now, since $y = e^{mx}$, we have

$$\frac{d^n y}{dx^n} = \frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}, \quad n = 1, 2, 3 \ldots$$

That is, $\frac{dy}{dx} = m e^{mx}$

$$\frac{d^2 y}{dx^2} = m^2 e^{mx}$$

$$\frac{d^3 y}{dx^3} = m^3 e^{mx} \text{ etc.}$$

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Thus from (1)

$m^n e^{mx} + P_1 m^{n-1} e^{mx} + P_2 m^{n-2} e^{mx} + \cdots + P_n e^{mx} = 0$

$\Rightarrow e^{mx}(m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n) = 0.$

$\Rightarrow m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0.$

Since $e^{mx} \neq 0$ —— (2)

This is an equation of nth degree in m.

Let $m_1, m_2 \cdots m_n$ be n roots of the equation (2). Then obviously $y = e^{m_1 x}$, $y = e^{m_2 x} \cdots$ $y = e^{m_n x}$ are solution of (1)

By theorem (I), the general or complete solution of (1) is given by

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

where $C_1, C_2, \cdots C_n$ are arbitrary constants.

Definition:- The equation (2) is called the Auxiliary equation of (1)

Formation of Auxiliary Equation. Let the differential equation with the right-hand member = 0 be

$$\frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = 0.$$

Or $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = 0$

Where $D^n = \frac{d^n}{dx^n}$

i.e, $f(D)y = 0$ where $f(D) =$

$$D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n.$$

We have shown in the preceeding article that the Auxiliary equation is

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$$

which is precisely $f(m) = 0$.

Thus the auxiliary equation can be written by replacing D by m.

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Method of finding complimentary function:-

We shall now consider the method of finding C.F. of a given equation, This will depend on the nature of roots of the auxiliary equation according as

(i) The roots are real and unequal.

(ii) The roots are real and equal.

(iii) The roots are conjugate complex and not repeated

(iv) The roots are conjugate complex and repeated.

CASE I:- Roots real and unequal

Let m_1 be a non-repeated root of the auxiliary equation $f(m)=0$. Then

$$f(m) = \{\phi(m)\}(m-m_1)$$

which $\Rightarrow f(D) = \{\phi(D)\}(D-m_1)$.

Hence any value of y that makes $(D-m_1)y=0$ must also ~~make~~ make

$$f(D)y = \{\phi(D)\}(D-m_1)y=0$$

Therefore solution of $(D-m_1)y=0$ is also a solution of $f(D)y=0$.

We now proceed to find the solution of $(D-m_1)y=0$.

$$\frac{dy}{dx}-m_1y=0 \Rightarrow \frac{dy}{y}=m_1dx$$

$$\Rightarrow \int\frac{dy}{y}=m_1\int dx \Rightarrow \log y=m_1x+\log c$$

$$\Rightarrow \log\frac{y}{c}=m_1x \Rightarrow y=ce^{m_1x}$$

Thus $y=ce^{m_1x}$ is a part of the complementary function.

Similarly if $m_1, m_2, m_3 --- m_n$ be the different roots of the auxiliary equation $f(m)=0$. The full value of the complementary function is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+---+C_ne^{m_nx}$$

where $C_1, C_2, --- C_n$ are arbitrary constants equal in number to the order of the equation.

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