10.06.2021

Mathematics Hons.
B. Sc. Part - I
Paper - I
Topic: Summation of Trigonometrical Series.

There are two methods for the summation of trigonometrical series; One is the method of difference and the other is what is called C + iS method.

(1) Sum Of Sines Of n Angles in A.P.

To find the sum of the sines of a series of angles which are in A.P.

Let the angles be

$\alpha, \alpha+\beta, \alpha+2\beta, \ldots, \{\alpha+(n-1)\beta\}$

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+\{n-1\}\beta)$

Now, we multiply each term of the series by

$2 \sin \frac{\text{Common diff.}}{2}$ and

express it as the difference of two cosines. Here c.d. $= \beta$.

we know that,

$$2 \sin \frac{\beta}{2} \cdot \sin(\alpha + k\beta)$$

$$= \cos\left(\alpha + \frac{2k-1}{2}\beta\right) - \cos\left(\alpha + \frac{2k+1}{2}\beta\right) \quad \text{--- (1)}$$

Now, putting $k = 0, 1, 2, 3 \dots (n-1)$ successively in (1), we get

$$2 \sin \frac{\beta}{2} \sin\alpha = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$2 \sin \frac{\beta}{2} \sin(\alpha + 1\beta) = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3}{2}\beta\right)$$

$$2 \sin \frac{\beta}{2} \sin(\alpha + 2\beta) = \cos\left(\alpha + \frac{3}{2}\beta\right) - \cos\left(\alpha + \frac{5}{2}\beta\right)$$

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$$2 \sin \frac{\beta}{2} \sin(\alpha + \overline{n-1}\beta) =$$

$$\cos\left(\alpha + \frac{2n-3}{2}\beta\right) - \cos\left(\alpha + \frac{2n-1}{2}\beta\right)$$

Adding these, we get

$$2 \sin \frac{\beta}{2} \cdot S = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{2n-1}{2}\beta\right)$$

$= 2\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}$

$\therefore S = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \sin\left(\alpha + \frac{n-1}{2}\beta\right)$

Cor: if we put $\beta = \frac{2\pi}{n}$, then we find that

$\sin\frac{n\beta}{2} = \sin\pi = 0.$

So that the sum of the series becomes $= 0$.

In other words,

$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \dots$ to n terms $= 0$

whatever be the value of α.

(ii) Sum of Cosines of n angles in A.P.

Let the angles be $\alpha, \alpha+\beta, \alpha+2\beta, \dots \{\alpha + (n-1)\beta\}$

Let $S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos(\alpha + \overline{n-1}\beta)$

Like before, we multiply

each term of the series by $2\sin \underline{c.d}$ and express it as the difference of two sines. Here $c.d = \beta$.

we know that

$$2\sin\frac{\beta}{2}\cos(\alpha+k\beta)$$

$$= \sin\left(\alpha+\frac{2k+1}{2}\beta\right) - \sin\left(\alpha+\frac{2k-1}{2}\beta\right) \quad \text{— (2)}$$

Now, putting $k=0,1,2,3 \dots (n-1)$ successively in (2), we get.

$$2\sin\frac{\beta}{2}\cos\alpha = \sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)$$

$$2\sin\frac{\beta}{2}\cos(\alpha+\beta) = \sin\left(\alpha+\frac{3}{2}\beta\right) - \sin\left(\alpha+\frac{1}{2}\beta\right)$$

$$2\sin\frac{\beta}{2}\cos(\alpha+2\beta) = \sin\left(\alpha+\frac{5}{2}\beta\right) - \sin\left(\alpha+\frac{3}{2}\beta\right)$$

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$$2\sin\frac{\beta}{2}\cos(\alpha+\overline{n-1}\,\beta) =$$

$$\sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin\left(\alpha+\frac{2n-3}{2}\beta\right)$$

Adding these, we get

$$2\sin\frac{\beta}{2}\cdot S = \sin\left(\alpha+\frac{2n-1}{2}\beta\right) - \sin\left(\alpha-\frac{\beta}{2}\right)$$

$$= 2\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}$$

$$S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Cor: if we put $\beta = \frac{2\pi}{n}$, then as before,

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \cdots \text{ to } n \text{ terms} = 0$$

Note: Putting $\alpha = \beta$ in both the formulae, we get

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \cdots + \sin n\alpha = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\sin\frac{n+1}{2}\alpha$$

and

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos n\alpha = \frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}\cdot\cos\frac{n+1}{2}\alpha.$$

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