12.06.2024

Mathematics Hons.
B. Sc. Part - I
Paper - I.

Topic: Summation Of Trigonometrical Series.

(1) Find the Sum of the Series

$$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \cdots + \sin^2(\alpha+\overline{n-1}\beta)$$

Soln:-

We use the following formula

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Hence the given series

$$= \frac{1-\cos 2\alpha}{2} + \frac{1-\cos(2\alpha+2\beta)}{2} + \frac{1-\cos(2\alpha+4\beta)}{2} + \cdots \text{ to } n \text{ terms}$$

$$= \frac{n}{2} - \frac{1}{2}\left[\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \cdots \text{ to } n \text{ terms}\right]$$

$$= \frac{n}{2} - \frac{1}{2}\cdot\frac{\sin n\beta}{\sin\beta}\cos(2\alpha+\overline{n-1}\beta)$$

(2) Sum the Series
$\sin\alpha \cdot \sin(\alpha+\beta) - \sin(\alpha+\beta) \cdot \sin(\alpha+2\beta) + \cdots$ to $2n$ terms.

Solⁿ: Let the required sum be S.

$\therefore S = \frac{1}{2}\left[2\sin\alpha \cdot \sin(\alpha+\beta) - 2\sin(\alpha+\beta)\sin(\alpha+2\beta) + \cdots \text{ to } 2n \text{ terms}\right]$

$= \frac{1}{2}\left[\{\cos(-\beta) - \cos(2\alpha+\beta)\} - \{\cos(-\beta) - \cos(2\alpha+3\beta)\} + \cdots \text{ to } 2n \text{ terms}\right]$

$= \frac{1}{2}\left[2n\cos\beta - \{\cos(2\alpha+\beta) - \cos(2\alpha+3\beta) + \cdots \text{ to } 2n \text{ terms}\right]$

$= n\cos\beta - \frac{1}{2}\left[\cos(2\alpha+\beta) + \cos(2\alpha+3\beta+\pi) + \cos(2\alpha+5\beta+2\pi) + \cdots \text{ to } 2n \text{ terms}\right]$

$= n\cos\beta - \frac{1}{2} \cdot \dfrac{\sin\frac{2n(2\beta+\pi)}{2}}{\sin\frac{2\beta+\pi}{2}} \times \cos\left\{2\alpha+\beta+\frac{2n-1}{2}(2\beta+\pi)\right\}$

$= n\cos\beta - \frac{1}{2} \cdot \dfrac{\sin n(2\beta+\pi)}{\cos\beta} \times \cos\left\{2\alpha+\beta+(2n-1)\left(\frac{\pi}{2}+\beta\right)\right\}.$

(3) Sum to n terms the series: $\sin\alpha \cdot \sin 2\alpha + \sin 2\alpha \cdot \sin 3\alpha + \sin 3\alpha \cdot \sin 4\alpha + \cdots$ and hence deduce the sum of the series

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$$

Solⁿ: We use the following formula:

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

Let, the required sum be S

$$\therefore S = \frac{1}{2}\Big[2\sin\alpha \sin 2\alpha + 2\sin 2\alpha \sin 3\alpha + 2\sin 3\alpha \sin 4\alpha + \cdots \text{ to } n \text{ terms.}\Big]$$

$$= \frac{1}{2}\Big[(\cos\alpha - \cos 3\alpha) + (\cos\alpha - \cos 5\alpha) + (\cos\alpha - \cos 7\alpha) + \cdots \text{ to } n \text{ terms}$$

$$= \frac{1}{2}\Big[n\cos\alpha - (\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \cdots \text{ to } n \text{ terms}$$

$$= \frac{n}{2}\cos\alpha - \frac{1}{2}\left[\frac{\sin n\cdot\frac{2\alpha}{2}}{\sin\frac{2\alpha}{2}}\cos\left\{3\alpha + \frac{n-1}{2}2\alpha\right\}\right]$$

$= \frac{n}{2}\cos\alpha - \frac{1}{2}\frac{\sin n\alpha \cdot \cos(n+2)\alpha}{\sin\alpha}$

eqn c can be simplified as

$= \left[\frac{n}{2}\cos\alpha \sin\alpha - \frac{1}{2}\sin n\alpha \cos(n+2)\alpha\right] / \sin\alpha.$

$= \left[\frac{n}{4}\, 2\sin\alpha\cos\alpha - \frac{1}{4}\cdot 2\sin n\alpha \cos(n+2)\alpha\right] / \sin\alpha.$

$= \frac{1}{4}\left[(n+1)\sin 2\alpha - \sin(2n+2)\alpha\right] / \sin\alpha$

Now, we proceed to find the sum of the series

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$

we have just proved that

$\sin\alpha \sin 2\alpha + \sin 2\alpha \sin 3\alpha + \dots$ ... + to n terms.

$= \frac{1}{4}\left[(n+1)\sin 2\alpha - \sin(2n+2)\alpha\right] / \sin\alpha.$

Dividing both sides by α^2 and expanding $\sin 2\alpha$ and $\sin(2n+2)\alpha$ in ascending powers of α, we get

$$2\cdot\frac{\sin\alpha}{\alpha}\cdot\frac{\sin 2\alpha}{2\alpha}+2\cdot 3\cdot\frac{\sin 2\alpha}{2\alpha}\cdot\frac{\sin 3\alpha}{3\alpha}+\cdots+n(n+1)\cdot\frac{\sin n\alpha}{n\alpha}\cdot\frac{\sin(n+1)\alpha}{(n+1)\alpha}$$

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$$=\frac{1}{4\alpha^2}\left[(n+1)\left\{2\alpha-\frac{(2\alpha)^3}{3!}+\cdots\right\}-\left\{(2n+2)\alpha-\frac{(2n+2)^3\alpha^3}{3!}+\cdots\right\}\right]\Big/\sin\alpha$$

$$=\frac{1}{4}\cdot\frac{\alpha}{\sin\alpha}\left[\frac{2^3}{3!}\left\{(n+1)^3-(n+1)\right\}-\frac{2^5\alpha^2}{5!}\left\{(n+1)^5-(n+1)\right\}+\cdots\right]$$

Now, taking the limit when $\alpha\to 0$, we get

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{1}{4}\cdot\frac{2^3}{3!}\left\{(n+1)^3-(n+1)\right\}\text{ as }\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1.$$

$$=\frac{1}{3}n(n+1)(n+2)$$

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