14.06.2021

Mathematics Hons.
B.Sc. Part. I
Paper. I
Topic: C+iS Method (Summation of Trigonometrical Series)

C+iS Method: The sum of many of the trigonometrical series can be found out by a general method which is called C+iS method. Let us suppose that we need to find out the sum of any one of the following series, whether finite or infinite.

$$C = a_0 \cos\alpha + a_1 \cos(\alpha+\beta) + a_2 \cos(\alpha+2\beta) + \cdots$$

and $$S = a_0 \sin\alpha + a_1 \sin(\alpha+\beta) + a_2 \sin(\alpha+2\beta) + \cdots$$

[The sine series is denoted by the first alphabet S and the cosine series is denoted by its first alphabet C.]

We multiply the sine-series by i and then add it to the cosine series. Then, $C+iS = a_0(\cos\alpha + i\sin\alpha) + a_1\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\} + a_2\{\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\} + \cdots$

$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \cdots \quad (1)$$

We shall see later on, that the series (1) is transformed in to one of the following forms.

(i) Series in G.P

(ii) Binomial series.

(iii) Exponential series.

(iv) Logarithmic series

(v) Sine or Cosine series.

(vi) Gregory's series.

Thus we express the sum of the series in the form $A+iB$ and then equating the real and imaginary parts, we get the values of C and S.

Use of Binomial Series;

We shall use the following expansions.

(i) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \cdots$

(ii) $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$

(iii) $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \cdots$

(iv) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(b) (i) Sum the following series to n terms:
$1 + x\cos\theta + x^2\cos 2\theta + \cdots$
Deduce the value when n increases infinitely when $|x| < 1$.
Soln:
Let $C = 1 + x\cos\theta + x^2\cos 2\theta + \cdots + x^{n-1}\cos(n-1)\theta$.
and $S = x\sin\theta + x^2\sin 2\theta + \cdots + x^{n-1}\sin(n-1)\theta$

$\therefore C + iS = 1 + x(\cos\theta + i\sin\theta) + x^2(\cos 2\theta + i\sin 2\theta) + \cdots + x^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$

$= 1 + xe^{i\theta} + x^2e^{i2\theta} + \cdots + x^{n-1}e^{i(n-1)\theta}$.

$= \dfrac{1-(xe^{i\theta})^n}{1-xe^{i\theta}}$, Since this series is in G.P whose c.r $= xe^{i\theta}$.

$= \dfrac{(1-x^ne^{in\theta})(1-xe^{-i\theta})}{(1-xe^{i\theta})(1-xe^{-i\theta})}$

$= \dfrac{1-xe^{-i\theta}-x^ne^{in\theta}+x^{n+1}e^{i(n-1)\theta}}{1-x(e^{i\theta}+e^{-i\theta})+x^2}$

$$= \frac{1 - x^n e^{in\theta} - x e^{-i\theta} + x^{n+1} e^{i(n-1)\theta}}{1 - 2x\cos\theta + x^2}$$ (4)

Numerator $= 1 - x^n(\cos n\theta + i\sin n\theta) - x.(\cos\theta - i\sin\theta) + x^{n+1}\{\cos(n-1)\theta + i\sin(n-1)\theta\}$

$\therefore$ The real part in the numerator $= 1 - x^n\cos n\theta - x\cos\theta + x^{n+1}\cos(n-1)\theta$

Now, equating the real part in (1), we get

$$C = \frac{1 - x^n\cos n\theta - x\cos\theta + x^{n+1}.\cos(n-1)\theta.}{1 - 2x\cos\theta + x^2}$$

This proves the first part.

Now, when $n \to \infty$, $x^n \to 0$ and $x^{n+1} \to 0$; $\because |x| < 1$.

$\therefore$ Sum to infinity

$$= \frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2}$$

(2) Find the Sum of the series
$1 + \cos\theta . \cos\theta + \cos^2\theta . \cos 2\theta +$
$\cos^3\theta \cos 3\theta + \cdots$ to ∞.

Sol:- Let $\cos\theta = x$. Then the series becomes $1 + x\cos\theta + x^2\cos 2\theta + \cancel{x^3\cos}$
$x^3 \cos 3\theta + \cdots$ to ∞

Let $C = 1 + x\cos\theta + x^2\cos 2\theta +$
$x^3\cos 3\theta + \cdots$ to ∞.

and $S = x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta$
$+ \cdots$ to ∞

$\therefore C + iS = 1 + x e^{i\theta} + x^2 e^{i2\theta} +$
$x^3 e^{i3\theta} + \cdots$ to ∞

$$= \frac{1}{1 - xe^{i\theta}} = \frac{1}{1 - xe^{i\theta}} \times \frac{1 - xe^{-i\theta}}{1 - xe^{-i\theta}}$$

$$= \frac{1 - x(\cos\theta - i\sin\theta)}{1 - x(e^{i\theta} + e^{-i\theta}) + x^2}$$

$$= \frac{(1 - x\cos\theta) + ix\sin\theta}{1 - 2x\cos\theta + x^2}$$

Hence equating the real part, we get

$$C = \frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2} = \frac{1 - \cos^2\theta}{1 - 2\cos^2\theta + \cos^2\theta}$$

On putting $x = \cos\theta$.

$$= \frac{\sin^2\theta}{1 - \cos^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} = 1.$$

Note: Equating the imaginary part, we get, $S = \dfrac{x\sin\theta}{1 - 2x\cos\theta + x^2} = \dfrac{\sin\theta\cos\theta}{\sin^2\theta} = \cot\theta$

Dr. Bikash Prasad
Asst. Prof.
Dept. of Maths.
D.K. College,
Dumraon.