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Mathematics Hons.

B. Sc. Part-I

Paper-I

Topic: C+iS Method. (Exp)

(H. Trigo.)

Q. Sum the Series

$$1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots \text{ad}\infty.$$

Soln: Let $C=1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\cdots$

and $S=-\frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta-\frac{1.3.5}{2.4.6}\sin 3\theta+\cdots$

Then $C+iS=1-\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1.3}{2.4}(\cos 2\theta+i\sin 2\theta)-\frac{1.3.5}{2.4.6}(\cos 3\theta+i\sin 3\theta)+\cdots$

$$=1-\frac{1}{2}e^{i\theta}+\frac{1.3}{2.4}e^{i2\theta}-\frac{1.3.5}{2.4.6}e^{i3\theta}+\cdots$$

$$=1-\frac{1}{2}x+\frac{1.3}{2.4}x^2-\frac{1.3.5}{2.4.6}x^3+\cdots;$$

On putting $e^{i\theta}=x$.

$$=(1+x)^{-1/2}$$

$$=(1+e^{i\theta})^{-1/2}=(1+\cos\theta+i\sin\theta)^{-1/2}$$

Now, we put $1+\cos\theta = r\cos\alpha$ and $\sin\theta = r\sin\alpha$

so that $r^2 = (1+\cos\theta)^2 + \sin^2\theta$

$$= 2(1+\cos\theta) = 2.2\cos^2\frac{\theta}{2}$$

$$\therefore r = 2\cos\frac{\theta}{2}$$

and $\tan\alpha = \dfrac{\sin\theta}{1+\cos\theta}$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}.$$

$$\therefore \alpha = \frac{\theta}{2}$$

Hence $C+iS = (r\cos\alpha + ir\sin\alpha)^{-1/2}$

$$= r^{-1/2}(\cos\alpha + i\sin\alpha)^{-1/2}$$

$$= r^{-1/2}(e^{i\alpha})^{-1/2}$$

$$= r^{-1/2}.e^{-i\alpha/2}$$

$$= \frac{1}{\sqrt{r}}\left\{\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right\}$$

Now equating the real part, we get

$$C = \frac{1}{\sqrt{r}}\cos\frac{\alpha}{2} = \frac{1}{\sqrt{2\cos\theta/2}}.\cos\frac{\theta}{4}.$$

Ex: Sum the Series.

$1+\frac{1}{2}\cos 2\theta-\frac{1}{2\cdot4}\cos 4\theta+\frac{1\cdot3}{2\cdot4\cdot6}\cos 6\theta-\cdots$ to ∞

Sol^n: Let $C=1+\frac{1}{2}\cos 2\theta-\frac{1}{2\cdot4}\cos 4\theta+\frac{1\cdot3}{2\cdot4\cdot6}\cos 6\theta-\cdots$

and $S=\frac{1}{2}\sin 2\theta-\frac{1}{2\cdot4}\sin 4\theta+\frac{1\cdot3}{2\cdot4\cdot6}\sin 6\theta-\cdots$

Then $C+iS=1+\frac{1}{2}e^{i2\theta}-\frac{1}{2\cdot4}e^{i4\theta}+\frac{1\cdot3}{2\cdot4\cdot6}e^{i6\theta}-\cdots$

$=1+\frac{1}{2}x-\frac{1}{2\cdot4}x^2+\frac{1\cdot3}{2\cdot4\cdot6}x^3-\cdots$

On putting $e^{i2\theta}=x$.

$=(1+x)^{1/2}=(1+e^{i2\theta})^{1/2}$

$=\{(1+\cos 2\theta)+i\sin 2\theta\}^{1/2}$

$=\{2\cos^2\theta+i2\sin\theta\cos\theta\}^{1/2}$

$=\sqrt{2\cos\theta}\,(\cos\theta+i\sin\theta)^{1/2}$

$=\sqrt{2\cos\theta}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$

Hence equating the real part, we get $C=\sqrt{2\cos\theta}\cos\frac{\theta}{2}$.

Q. ~~Sum to n~~

find the Sum of the series $\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \cdots + n\cos n\theta$

Solⁿ:- Let,

$C = \cos\theta + 2\cos 2\theta + 3\cos 3\theta + \cdots + n\cos n\theta$

and $S = \sin\theta + 2\sin 2\theta + 3\sin 3\theta + \cdots + n\sin n\theta$

$\therefore C + iS = (\cos\theta + i\sin\theta) + 2(\cos 2\theta + i\sin 2\theta) + 3(\cos 3\theta + i\sin 3\theta) + \cdots + n(\cos n\theta + i\sin n\theta)$

$= e^{i\theta} + 2e^{i2\theta} + 3e^{i3\theta} + \cdots + ne^{in\theta}$

$= e^{i\theta} + 2e^{i2\theta} + 3e^{i3\theta} + \cdots + ne^{in\theta}$

$= x + 2x^2 + 3x^3 + \cdots + nx^n$

where $x = e^{i\theta}$.

Now, first of all we find the sum of the series on the R.H.S, namely $x + 2x^2 + 3x^3 + \cdots + nx^n$ which is an arithmetic geometric series.

Let $T = x + 2x^2 + 3x^3 + \cdots + nx^n$

Then $Tx = x^2 + 2x^3 + \cdots + (n-1)x^n + nx^{n+1}$

On Substraction,

$T(1-x) = (x + x^2 + x^3 + \cdots + x^n) - nx^{n+1}$

$$= x(1 + x + x^2 + \dots + x^{n-1}) - nx^{n+1}$$

$$= \frac{x(1-x^n)}{1-x} - nx^{n+1}$$

$$= \frac{x - x^{n+1} - nx^{n+1} + nx^{n+2}}{1-x}.$$

$$T = \frac{x - x^{n+1} - nx^{n+1} + nx^{n+2}}{(1-x)^2}$$

$$= \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

$$\Rightarrow C + iS = \frac{e^{i\theta} - (n+1)e^{i(n+1)\theta} + ne^{i(n+2)\theta}}{(1-e^{i\theta})^2} \quad (1)$$

Now, $(1-e^{i\theta})^2 = (1-\cos\theta - i\sin\theta)^2$

$$= \left(2\sin^2\frac{\theta}{2} - i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2$$

$$= 4\sin^2\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^2$$

$$= 4\sin^2\frac{\theta}{2}\left[-i\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^2$$

$$= -4\sin^2\frac{\theta}{2}\left(e^{i\theta/2}\right)^2 = -4\sin^2\frac{\theta}{2}\,e^{i\theta}$$

Substituting this value in (1), we get

$$C + iS = \frac{e^{i\theta} - (n+1)e^{i(n+1)\theta} + ne^{i(n+2)\theta}}{-4\sin^2\frac{\theta}{2}\cdot e^{i\theta}}$$

$$= \frac{1}{4\sin^2\frac{\theta}{2}}\left[1 - (n+1)e^{in\theta} + ne^{i(n+1)\theta}\right]$$

$$= -\frac{1}{4\sin^2\frac{\theta}{2}}\left[1 - (n+1)(\cos n\theta + i\sin n\theta) + n\{\cos(n+1)\theta + i\sin(n+1)\theta\}\right]$$

Hence equating the real part from both sides, we get

$$C = -\frac{1}{4\sin^2\frac{\theta}{2}}\left[1 - (n+1)\cos n\theta + n\cos(n+1)\theta\right]$$

Which is the required form.

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