15.06.2021.

Mathematics Hons.
B. Sc. Part-II
Paper-IV
Topic: Product of three and four vectors.

Theorem:- Establish a necessary and sufficient condition that the three non-parallel non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ be coplanar is $[\vec{a}\,\vec{b}\,\vec{c}]=0$

Proof:- Necessary Condition:

Let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors. Since $\vec{b}\times\vec{c}$ represents a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$, $\vec{a}$ lies in that plane so $\vec{b}\times\vec{c}$ is also perpendicular to $\vec{a}$. Thus their scalar triple product $a\cdot(\vec{b}\times\vec{c})=0$. Hence when three vectors are coplanar then their scalar triple product is zero.

Sufficient Condition:

Let $[\vec{a}\,\vec{b}\,\vec{c}]=0$.

i.e, $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ then $\vec{b}\times\vec{c}$ is perpendicular to $\vec{a}$. Since $\vec{b}\times\vec{c}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. and thus $\vec{a}$ should also lie in the plane of $\vec{b}$ and $\vec{c}$. Hence $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

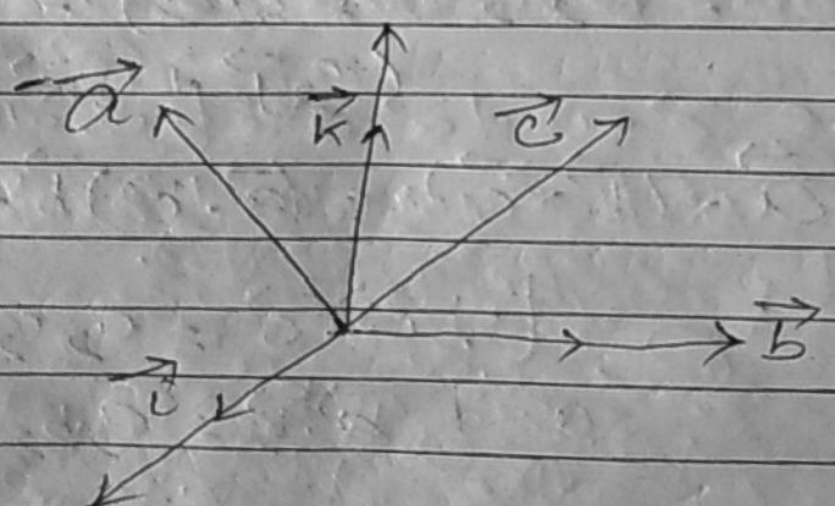
Theorem: Prove that

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Or,

Write geometrical interpretation of vector triple product.

Proof: Let us take any arbitrary vector $\vec{a}$ a, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, vector $\vec{b}$ along $\vec{j}$ as $\vec{b} = b_2\vec{j}$ and $\vec{c}$ in the plane of $\vec{j}$ and $\vec{k}$ as $\vec{c} = c_2\vec{j} + c_3\vec{k}$.



Then we have

$\vec{b} \times \vec{c} = b_2\vec{j} \times (c_2\vec{j} + c_3\vec{k})$

$\Rightarrow \vec{b} \times \vec{c} = b_2 c_3 \vec{j} \times \vec{k}$ $(\because \vec{j} \times \vec{j}) = 0$

$= b_2 c_3 \vec{i}$ $[\because \vec{j} \times \vec{k} = \vec{i}]$

Thus $\vec{a} \times (\vec{b} \times \vec{c})$

$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times b_2 c_3 \vec{i}$

$= 0 + a_2 b_2 c_3 \vec{j} \times \vec{i} + a_3 b_2 c_3 \vec{k} \times \vec{i}$

$[\because \vec{i} \times \vec{i} = 0]$

$\Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = -a_2 b_2 c_3 \vec{k} + a_3 b_2 c_3 \vec{j}$

$= a_2 c_2 b_2 \vec{j} + a_3 b_2 c_3 \vec{j} - a_2 b_2 c_3 \vec{k} - a_2 c_2 b_2 \vec{j}$

[$\because$ Additing and substracting by $a_2 c_2 b_2 \vec{j}$]

$= (a_2 c_2 + a_3 c_3) b_2 \vec{j} - a_2 b_2 (c_2\vec{j} + c_3\vec{k})$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

[$\because \vec{a} \cdot \vec{c} = a_2 c_2 + a_3 c_3$

and $\vec{a} \cdot \vec{b} = a_2 b_2$]

<u>Theorem</u>: Prove that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) =$$

$$(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

<u>Proof</u>:- L.H.S $= (\vec{a} \times \vec{b}) (\vec{c} \times \vec{d})$

Let $\vec{m} = \vec{c} \times \vec{d}$, then

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \cdot \vec{m}$$

$$= \vec{a} \cdot (\vec{b} \times \vec{m})$$

[∵ Position of dot and cross can be interchanged]

$$= \vec{a} \cdot [\vec{b} \times (\vec{c} \times \vec{d})]$$

$$= \vec{a} \cdot [(\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d}]$$

$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

Theorem: Prove that
$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

Proof:- Let $\vec{a}\times\vec{b} = \vec{m}$

we have $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{m}\times(\vec{c}\times\vec{d})$

$= (\vec{m}\cdot\vec{d})\vec{c} - (\vec{m}\cdot\vec{c})\vec{d}$

$= [(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c} - [\vec{c}\cdot(\vec{a}\times\vec{b})]\vec{d}$

$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

Theorem: Show that
$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ iff $(\vec{c}\times\vec{a})\times\vec{b} = 0$.
Or, if $\vec{c}$ and $\vec{a}$ are collinear.

Proof:- Given $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$

$\Leftrightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

$\Leftrightarrow -(\vec{a}\cdot\vec{b})\vec{c} = -(\vec{b}\cdot\vec{c})\vec{a}$

[$\because$ Eliminate $(\vec{a}\cdot\vec{c})\vec{b}$ from both]

$\Leftrightarrow (\vec{a}\cdot\vec{b})\vec{c} = (\vec{b}\cdot\vec{c})\vec{a}$

$\Leftrightarrow (\vec{a}\cdot\vec{b})\vec{c} = (\vec{c}\cdot\vec{b})\vec{a}$

$\Leftrightarrow (\vec{c}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c} = 0.$

$\Leftrightarrow (\vec{c}\times\vec{a})\times\vec{b} = 0.$

When $(\vec{c}\times\vec{a})\times\vec{b} = 0$ then either
$\vec{c}\times\vec{a} = 0$ or $\vec{b} = 0$.

But $\vec{b} \neq 0$ then we must have
$\vec{c}\times\vec{a} = 0$.

i.e, $\vec{c}$ and $\vec{a}$ are parallel. Hence
$\vec{c}$ and $\vec{a}$ are collinear.

Dr. Birendra Prasad
Asst. Prof.
Dept. of Maths.
D.K. College,
Dumraon.