17.06.2021.

Mathematics Hons.

B. Sc. Part - II

Paper - IVth

Topic: Product of three and four vectors

(1.) Prove that

$$\vec{i}\times(a\times\vec{i}) + \vec{j}\times(a\times\vec{j}) + \vec{k}\times(a\times\vec{k}) = 2\vec{a}$$

Solⁿ:- We know that

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

using this formula in L.H.S. we have L.H.S =

$$(\vec{i}.\vec{i})\vec{a} - (\vec{i}.\vec{a})\vec{i} + (\vec{j}.\vec{j})\vec{a} - (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{k})\vec{a} - (\vec{k}.\vec{a})\vec{k}$$

$$= \vec{a} - (\vec{i}.\vec{a})\vec{i} + \vec{a} - (\vec{j}.\vec{a})\vec{j} + \vec{a} - (\vec{k}.\vec{a})\vec{k} \quad [\because \vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1]$$

$$= 3\vec{a} - (\vec{i}.\vec{a})\vec{i} - (\vec{j}.\vec{a})\vec{j} - (\vec{k}.\vec{a})\vec{k} \quad \text{--- (1)}$$

Let, $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ --- (2)

Then $(\vec{i}.\vec{a})\vec{i} = [\vec{i}.(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})]\vec{i}$

$= [a_1 + 0 + 0]\vec{i} \quad [\because \vec{i}.\vec{j} = \vec{i}.\vec{k} = 0]$

$= a_1\vec{i}$

in the same way, we have

$(\vec{j}\cdot\vec{a})\vec{j} = a_2\vec{j}$ and $(\vec{k}\cdot\vec{a})\vec{k} = a_3\vec{k}$

By (1)

L.H.S $= 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$= 3\vec{a} - \vec{a}$ [using (2)]

$= 2a.$

(2) Prove that

$[\vec{a}\times\vec{b}, \vec{c}\times\vec{d}, \vec{e}\times\vec{f}]$

$= [\vec{a}\,\vec{b}\,\vec{d}][\vec{c}\,\vec{e}\,\vec{f}] - [\vec{a}\,\vec{b}\,\vec{c}][\vec{d}\,\vec{e}\,\vec{f}]$

Sol:-

L.H.S $= [\vec{a}\times\vec{b}, \vec{c}\times\vec{d}, \vec{e}\times\vec{f}]$

$= (\vec{a}\times\vec{b})[(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})]$

$= (\vec{a}\times\vec{b})\{[\vec{c}\,\vec{e}\,\vec{f}]\vec{d} -$

$[\vec{d}\,\vec{e}\,\vec{f}]\vec{c}\}$

$= [\vec{a}\,\vec{b}\,\vec{d}][\vec{c}\,\vec{e}\,\vec{f}] -$

$[\vec{a}\,\vec{b}\,\vec{c}][\vec{d}\,\vec{e}\,\vec{f}]$

$=$ R.H.S.

(3) Prove that $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ and deduce that $[\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}] = [abc]^2$

Sol^n: We know from product of four vectors that,

$$(\vec{b} \times \vec{c})(\vec{c} \times \vec{a}) = [\vec{b}\,\vec{c}\,\vec{a}]\,\vec{c} - [\vec{b}\,\vec{c}\,\vec{c}]\,\vec{a}$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\,\vec{c} - 0$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\,\vec{c}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$$

$$[\because [\vec{b}\,\vec{c}\,\vec{c}] = 0]$$

2nd part:

$$[\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}]$$

$$= (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) \cdot \vec{a} \times \vec{b}$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\,\vec{c} \cdot \vec{a} \times \vec{b}$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\,[\vec{c} \cdot \vec{a} \times \vec{b}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{c}\,\vec{a}\,\vec{b}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]^2$$

4. Find the volume of the parallelopiped whose edges are represented by $\vec{i} + 2\vec{j} + 3\vec{k}$, $3\vec{i} + 7\vec{j} - 4\vec{k}$ and $\vec{i} - 5\vec{j} + 3\vec{k}$

Soln. We know that volume of parallelopiped $= \vec{a} \cdot (\vec{b} \times \vec{c})$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot [(3\vec{i} + 7\vec{j} - 4\vec{k}) \times (\vec{i} - 5\vec{j} + 3\vec{k})]$$

$$= \vec{i} + 2\vec{j} + 3\vec{k} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot [\vec{i}(21-20) - \vec{j}(9+4) + \vec{k}(-15-7)]$$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (\vec{i} - 13\vec{j} - 22\vec{k})$$

$$= 1 - 26 - 66 = -91$$

$$= +91 \text{ (Cubic units)}$$

[As volume can not be negative]

(5) If $\vec{a}\ \vec{b}\ \vec{c}$ be three unit vectors such that

$$\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$$

Find the angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, where $\vec{b}$ and $\vec{c}$ are non parallel.

Solⁿ!

Given that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c} = \frac{1}{2}\vec{b}$$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - \frac{1}{2}\vec{b} = (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$\Rightarrow \left(\vec{a}\cdot\vec{c} - \frac{1}{2}\right)\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c} = 0.$$

This is possible only when

$\vec{a}\cdot\vec{c} - \frac{1}{2} = 0$ and $\vec{a}\cdot\vec{b} = 0$

[$\because$ $\vec{b}$ and $\vec{c}$ are not parallel]

When $\vec{a}\cdot\vec{c} - \frac{1}{2} = 0 \Rightarrow \vec{a}\cdot\vec{c} = \frac{1}{2}$

$$\Rightarrow 1\cdot 1\cdot \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ.$$

[When $\vec{a}\cdot\vec{b} = 0$ then $\vec{a}$ is $\perp$ to $\vec{b}$ then angle between them say $\phi = 90^\circ$

Thus $\vec{a}$ makes angles 60° and 90° respectively with $\vec{b}$ and $\vec{c}$]

forwarded.

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