

18.06.2021.

Mathematics Honours & Sub.
B.Sc. Part-I

Paper-I

Topic: Convex Set (L.P)

Properties of Convex Set

Theorem! A Hyperplane is a Convex Set.

Proof! Let H be the hyperplane defined by

$$H = \{x: c_1x_1 + c_2x_2 + \dots + c_nx_n = a\}$$

Let $y \in H$, $z \in H$, so

$$c_1y_1 + c_2y_2 + \dots + c_ny_n = a \quad \text{--- (1)}$$

$$\text{and } c_1z_1 + c_2z_2 + \dots + c_nz_n = a \quad \text{--- (2)}$$

where $y = (y_1, y_2, y_3, \dots, y_n)$

and $z = (z_1, z_2, z_3, \dots, z_n)$

Now we are to prove that every point of the line segment joining y and z is in H .
for this, let S be any point on the line segment joining y and z , then

$$S = (S_1, S_2, S_3, \dots, S_n)$$

$$= \lambda y + (1-\lambda)z, \text{ where } 0 \leq \lambda \leq 1.$$

$$\text{Now, } C_1 C_1 + C_2 C_2 + C_3 C_3 + \dots + C_n C_n.$$

$$= C_1 \{ \lambda y_1 + (1-\lambda) z_1 \} + C_2 \{ \lambda y_2 + (1-\lambda) z_2 \} + \dots + C_n.$$

$$\{ \lambda y_n + (1-\lambda) z_n \}$$

$$= \lambda (C_1 y_1 + C_2 y_2 + \dots + C_n y_n) +$$

$$(1-\lambda) (C_1 z_1 + C_2 z_2 + \dots + C_n z_n)$$

$$= \lambda a + (1-\lambda) a; \text{ from (1) \& (2)}$$

$$= a$$

$$\therefore S \in H.$$

Hence the hyperplane H is a Convex Set.

Theorem 2: Every line Segment is a Convex Set.

Proof: Let S be the set of points on the line segment joining the points x and y . Then by definition,

$$S = \{ a : a = \lambda x + (1-\lambda) y, 0 \leq \lambda \leq 1 \}$$

if $a_1, a_2 \in S$ then

$$a_1 = \lambda_1 x + (1-\lambda_1) y, 0 \leq \lambda_1 \leq 1.$$

$$\text{and } a_2 = \lambda_2 x + (1-\lambda_2) y, 0 \leq \lambda_2 \leq 1.$$

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We have to show that

$$b = k a_1 + (1-k) a_2 \in S, 0 \leq k \leq 1.$$

$$\text{Now } b = \lambda [\lambda_1 x + (1-\lambda_1) y] + (1-k) [\lambda_2 x + (1-\lambda_2) y]$$

$$= x [k\lambda_1 + (1-k)\lambda_2] + y [(1-\lambda_1)k + (1-k)(1-\lambda_2)]$$

Here, we find that

$$k\lambda_1 + (1-k)\lambda_2 \geq 0 \text{ and}$$

$$(1-\lambda_1)k + (1-k)(1-\lambda_2) \geq 0.$$

Because

$$0 \leq k \leq 1, 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1.$$

$$\text{and } [k\lambda_1 + (1-k)\lambda_2] + [(1-\lambda_1)k + (1-k)(1-\lambda_2)]$$

$$= k\lambda_1 + \lambda_2 - k\lambda_2 + k - k\lambda_1 + 1 - \lambda_2 - k + k\lambda_2 = 1$$

$$\therefore b \in S.$$

Thus S is a Convex Set.

Theorem: The Set of all Convex Combinations of a finite number of points in \mathbb{R}^n is a Convex Set.

Or,

A Convex polyhedron is a Convex Set.

Proof: Let S be the set of

all convex combinations
(i.e., convex polyhedron) of
a finite number of points
 $x_1, x_2, x_3, \dots, x_n$. For $x \in S$,
let,

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_n x_n.$$

where λ_i 's ≥ 0 and $\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n = 1$.

Let $y \in S$ and $z \in S$. Then we
are to prove that the line
segment joining y and z lies in
 S .

Let S be a point on the
line segment joining y and z .
 $\therefore S = \lambda y + (1-\lambda)z, 0 \leq \lambda \leq 1$.

As $y \in S$,

$$y = \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_n x_n$$

where μ_i 's ≥ 0 and $\mu_1 + \mu_2 + \mu_3 + \dots + \mu_n = 1$.

As $z \in S$,

$$z = \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_n x_n$$

where μ_i 's ≥ 0 and $\mu_1 + \mu_2 + \mu_3 + \dots + \mu_n = 1$.

$$\therefore S = \lambda y + (1-\lambda)z$$

$$= \lambda [h_1 x_1 + h_2 x_2 + \dots + h_n x_n] + (1-\lambda) [u_1 x_1 + u_2 x_2 + \dots + u_n x_n]$$

$$= [\lambda h_1 + (1-\lambda)u_1] x_1 +$$

$$[\lambda h_2 + (1-\lambda)u_2] x_2 + \dots +$$

$$[\lambda h_n + (1-\lambda)u_n] x_n$$

$$= m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots + m_n x_n$$

where $m_i = [\lambda h_i + (1-\lambda)u_i] x_i \geq 0$

for $h_i \geq 0$, $u_i \geq 0$ and $0 \leq \lambda \leq 1$.

and $m_1 + m_2 + m_3 + \dots + m_n$

$$= [\lambda h_1 + (1-\lambda)u_1] + [\lambda h_2 + (1-\lambda)u_2] + \dots + [\lambda h_n + (1-\lambda)u_n]$$

$$= \lambda (h_1 + h_2 + h_3 + \dots + h_n)$$

$$+ (1-\lambda) (u_1 + u_2 + \dots + u_n)$$

$$= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$$

$\therefore S \in S$.

Thus S is a convex set.

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