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Mathematics Home

B. Sc. Part-I

Paper-I

Topic: Convex Set (L.P)

Theorem! The Set of feasible solutions to a linear programming problem is a Convex Set!

OR,
Prove that the set of solutions of the equations and inequalities

$$\sum_{j=1}^r a_{ij}x_j \leq \text{or } \geq b_i,$$

$$i = 1, 2, 3, \dots, m.$$

$x_j \geq 0, j = 1, 2, 3, \dots, r$ is a Convex Set.

Proof.

Let S be the set of solutions of the $(m+r)$ equations and inequalities

$$\sum_{j=1}^r a_{ij}x_j \leq b_i, \quad i = 1, 2, 3, \dots, m \quad \text{--- (1)}$$

$$x_j \geq 0, \quad j = 1, 2, 3, \dots, r \quad \text{--- (2)}$$

Let $y \in S, z \in S$. Let w be

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any points on the line
segment joining y and z . Then

$$w = \lambda y + (1-\lambda)z; \quad 0 \leq \lambda \leq 1$$

Now,

$$\sum_{j=1}^r a_{ij} w_j = \sum_{j=1}^r a_{ij} [\lambda y_j + (1-\lambda)z_j]$$

$$= \lambda \sum_{j=1}^r a_{ij} y_j + (1-\lambda) \sum_{j=1}^r a_{ij} z_j$$

$$\leq \lambda b_i + (1-\lambda)b_i$$

$$= b_i; \quad i = 1, 2, \dots, m.$$

Because $y \in S$ and $z \in S$
satisfy (1) and (2)

$$\sum_{j=1}^r a_{ij} w_j \leq b_i; \quad i = 1, 2, \dots, m$$

$$\text{Again } w_j = \lambda y_j + (1-\lambda)z_j$$

$$\sum_{j=1}^r \lambda \cdot 0 + (1-\lambda)0 = 0;$$

$$j = 1, 2, \dots, r.$$

$\therefore w$ satisfies (1) and (2).

So $w \in S$.

Thus S contains all points of the line joining any points in it. Hence S is a convex set.

Example

Prove that the sphere is a convex set.

Solⁿ. We take the equation of the sphere as

$$x^2 + y^2 + z^2 = a^2 \quad \text{--- (1)}$$

and let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be any two points in the sphere

$$\left. \begin{aligned} \therefore x_1^2 + y_1^2 + z_1^2 &\leq a^2 \\ \text{and } x_2^2 + y_2^2 + z_2^2 &\leq a^2 \end{aligned} \right\} \text{--- (2)}$$

The convex combination of v_1 and v_2 is given by:

$$v = \lambda v_1 + (1-\lambda)v_2, \quad 0 \leq \lambda \leq 1 \quad \text{--- (3)}$$

Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$ and $v = (\alpha, \beta, \gamma)$

$$\left. \begin{aligned} \alpha &= \lambda x_1 + (1-\lambda)x_2 \\ \beta &= \lambda y_1 + (1-\lambda)y_2 \\ \gamma &= \lambda z_1 + (1-\lambda)z_2 \end{aligned} \right\} \text{--- (4)}$$

α, β, γ means the co-ordinates of any point on the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by (4)

Now,

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= [\lambda x_1 + (1-\lambda)x_2]^2 + [\lambda y_1 + (1-\lambda)y_2]^2 + [\lambda z_1 + (1-\lambda)z_2]^2 \\ &= \lambda^2(x_1^2 + y_1^2 + z_1^2) + (1-\lambda)^2(x_2^2 + y_2^2 + z_2^2) \\ &\quad + 2\lambda(1-\lambda)(x_1x_2 + y_1y_2 + z_1z_2) \end{aligned}$$

$$\leq \lambda^2 a^2 + (1-\lambda)^2 a^2 + 2\lambda(1-\lambda) \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$\sqrt{x_2^2 + y_2^2 + z_2^2}$$

(Cauchy inequality)

$$\begin{aligned} &\leq \lambda^2 a^2 + (1-\lambda)^2 a^2 + 2\lambda(1-\lambda)a \\ &= a^2 [\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)] \\ &= a^2 [\lambda + (1-\lambda)]^2 = a^2 \end{aligned}$$

Thus $\alpha^2 + \beta^2 + \gamma^2 \leq a^2$
 \Rightarrow any point on the line segment belongs to the set.
 But the point is arbitrary.
 Hence the line segment represented by (B) belongs to the set.

Hence a sphere is a convex set.

~~Ex~~ Show that every extreme point of the convex set of all feasible solutions of the system $Ax = b$ is a basis feasible solution.

Solⁿ: Suppose x_1 a n -component vector, is a basis feasible solution of $Ax = b$ and $x = [x_3, 0]$ where 0 is a vector of n -components

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and x_3 is the vector of m basis feasible. Also if Bx the matrix of vectors associated to basis vectors in the basis feasible solution, then

$$Ax = b, \quad Bx_B = b \quad \text{--- (1)}$$

Now, we are to show that x is an extreme point.

Suppose x is not an extreme point, then there exists x_1 and x_2 in the set X (Convex set of all feasible solution of $Ax = b$) s.t. $x = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$ --- (2)

Now, let $x_1 = [u_1, v_1]$, $x_2 = [u_2, v_2]$ where u_1 and u_2 respectively are the vectors of values in x_1 and x_2 i.e. the basis variables of x and similarly v_1 and v_2 are the vectors of values of those variables which are zero in x . Then (2) shows that

$$0 = \lambda v_1 + (1-\lambda)v_2, \quad 0 < \lambda < 1.$$

Since $\lambda > 0$, $1-\lambda > 0$ and also $v_1 \geq 0$, $v_2 \geq 0$

This relation can only be satisfied when $v_1 = 0$ and $v_2 = 0$

With these values of v_1 and v_2 , we have $x_1 = [u_1, 0]$, $x_2 = [u_2, 0]$. Therefore $Ax_1 = Bu_1 = b$ and $Ax_2 = Bu_2 = b$

$$\therefore b - Bx_B = Bu_1 = Bu_2$$

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$$\Rightarrow x_B = u_1 = u_2$$

$$\therefore x = [x_B, 0]$$

$$= [u_1, 0]$$

$$= [u_2, 0]$$

$$\text{Or } x = x_1 = x_2$$

It shows that there do not exist two points x_1, x_2 different than x , of which x is a convex combination.

Hence, by definition,

x is an extreme point.

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