On dated: 25/06/2021

Mathematics Hons.
B. Sc. Part - III
Paper - V^{th}
Topic: Limits (Real Analysis)

Inner limits:

$\lim_{y \to b} f(x,y) = g(x) = (x)\ (\epsilon > 0)$

$(\exists V(b)[x \in S \text{ and } y \in V(b)$

$\Rightarrow |f(x,y) - \phi(x)| < \epsilon|$

$\lim_{x \to a} f(x,y) = h(y) \equiv y(\epsilon > 0)\ (\exists V(a))$

$|y \in T \text{ and } x \in V(a)$

$\Rightarrow |f(x,y) - h(y)| < \epsilon|$

Uniform limit: Let us consider

$\lim_{n \to a} f(x,y) = g(y),\ y \in T$

Therefore for each fixed $y \in T$, corresponding to any $\epsilon > 0$, there exists $\delta(y)$ S.t

$|f(x,y) - g(y)| < \epsilon$ provided $|x-a| < \delta(y)$

This can be expressed Symbolically as

$(\epsilon > 0)\ (\exists \partial(y))\ [|f(x,y) - g(y)| < \epsilon,$ provided $|x-a| < \delta]$

Here δ depends upon ϵ as well as upon y.

if in certain cases δ depends upon ϵ only but not upon y i.e. $(\epsilon > 0)\ (\exists\, \delta(y))\ [\,|f(x,y) - g(y)| < \epsilon$ provided $|x - a| < \delta\,]$

then we say that

$$\lim_{x \to a} f(x,y) = g(y)$$

uniformly over T

$\lim_{y \to b} f(x,y) = g(x)$ uniformly over S if $(\epsilon > 0)(\exists\, V(b))(x)$

$$[\,y \in V(b) \Rightarrow |f(x,y) - g(x)| < \epsilon\,]$$

Pointwise limit:

$$\lim_{y \to b} f(x,y) = g(x) \text{ pointwise}$$

if $(\epsilon > 0)\ (\exists\, V(b)(x)\ [\,y \in V(b) \Rightarrow |f(x,y) - g(x)| < \epsilon\,]$

Continuity:

Let $f(x,y)$ be a real valued function of two real variables x, y where (x,y) varies over $S \times T \subseteq R^2$ and let $a \in S$, $b \in T$. Then the domain of S is $S \times T$ and $(a,b) \in S \times T = \text{dom} f$.

Now, f is said to be continuous at a point (a,b) of its domain if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

In other hand, a function f is continuous at a point (a,b) of its domain if given $\varepsilon > 0$, however small, $\exists$ a δ neighbourhood V of (a,b) s.t

$|f(x,y) - f(a,b)| < \varepsilon$ for all $(x,y) \in V$

i.e, $|f(x,y) - f(a,b)| < \varepsilon$ for all (x,y) satisfying $|x-a| < \delta$, $|y-b < \delta$.

A function which is not continuous at a point is said to be discontinuous at the point.

MOORE-OSGOOD THEOREM

Statement!

if f (x,y) be a real valued function of two variables x, y where $(x,y) \in A \times B$ and if $a \in \bar{A}$ the closure of A, $b \in \bar{B}$, the closure of B and if

(i) $\lim_{x\to a} f(x,y)$ exists for every y for which f(x,y) is defined

(ii) $\lim_{y\to a} f(x,y)$ exists for every x for which f(x,y) is defined and

(iii) either of the above two limits is uniform, then the double limit the repeated limits exist and are all equal.

Proof:- Since $\lim\limits_{x \to a} f(x,y)$ exists.

Suppose,

$$\lim_{x \to a} f(x,y) = g(y) \quad \text{--- (1)}$$

Again,

$$\lim_{y \to b} f(x,y) \text{ exists}$$

So let

$$\lim_{y \to b} f(x,y) = h(x) \quad \text{--- (2)}$$

Also suppose the limit (2) in uniform with respect to x which means that given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,y_1) - f(x,y)| < \epsilon \quad \text{--- (3)}$$

for all values of y and y_1 satisfying $|y-b| < \delta$ and $|y_1-b| < \delta$ and for all x.

Now we suppose $x \to a$ is in (3),

We have from (4)

$$|g(y_1) - g(y)| < \varepsilon \quad \text{--- (4)}$$

for all values of y and y_1 for which $|y-b| < \delta$ and $|y_1 - b| < \delta$.

But from (4), we conclude that $\lim_{y\to b} g(y)$ exists

Suppose $\lim_{y\to b} g(y) = l$ --- (5)

Again if we let $y_1 \to b$ in (4), we have $|l - g(y)|$ --- (6)

for which $|y-b| < \delta$

Again suppose $y_1 \to b$ in (3), then in view of (2), we get

$$|h(x) - f(x,y)| < \varepsilon \quad \text{--- (7)}$$

for all y for which $|y-b| < \delta$ and for all x.

Now, $|l - h(x)| = |l - g(y) + g(y) - f(x,y) + f(x,y) - h(x)| \leq |l - g(y)| + |g(y) - f(x,y)| + |f(x,y) - h(x)|$

$< \varepsilon + \varepsilon + \varepsilon$ from (6), (1) and (7)

$$\Rightarrow \lim_{x\to a} h(x) = l \quad \text{--- (8)}$$

Now, from (5) and (8) we see that

$$\lim_{y\to b} \lim_{x\to a} = \lim_{x\to a} \lim_{y\to b} f(x,y)$$

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