Dated On: 26th June 21.

Mathematics Hons.

B. Sc. Part - III

Paper - Vth

Topic: Repeated limits (Exp).

(Real Analysis)

~~Ex.~~ (1) Consider the function

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad x^2+y^2 \neq 0.$$

$$= 0, \text{ if } x^2+y^2 = 0$$

Solⁿ:- For repeated limit

$$\lim_{y\to 0} \lim_{x\to 0} \frac{xy}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{0}{x^2}$$

$$= 0.$$

and

$$\lim_{x\to 0} \lim_{y\to 0} \frac{xy}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{0}{y^2}$$

$$= \lim_{x\to 0} 0 = 0.$$

Thus we see that the two repeated limits exist and are equal at the origin. But the double limit (Simultaneous limit) does not exist at the origin for if we put $y = mx$ $(m \neq 0)$

$$\frac{xy}{x^2+y^2} = \frac{x \cdot mx}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

and this will have different values for different m.

Thus $x \to 0$ (and consequently $y \to 0$), the double limit or simultaneous limit does not exist.

~~Ex. 2:~~ The function defined by

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

The repeated limits exist but the double limit which is the simultaneous limit does not exist when $(x,y) \to (0,0)$

Soln:- Here,

$$\lim_{y \to 0} \lim_{x \to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

$$= \lim_{y \to 0} \frac{0}{0+(0-y)^2}$$

$$= \lim_{y \to 0} 0 = 0.$$

and $\lim_{x\to 0} \lim_{y\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2}$

$$= \lim_{x\to 0} \frac{0}{0+x^2}$$

$$= 0.$$

Hence the two repeated limits exists and are equal.

For simultaneous or double limit.

Putting $y = x$.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0} \frac{x^2x^2}{x^2x^2+(x-x)^2}$$

$$= \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

Which is different from the common value of the two repeated limits.

Thus $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

(2) Find the double limit and repeated limits for the function

$$f(x,y) = \begin{cases} xy \frac{(x^2-y^2)}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$$

when $(x,y) \to (0,0)$

Solution!

For repeated limits

$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy(x^2-y^2)}{x^2+y^2}$$

$$=\lim_{y\to 0} 0 = 0.$$

and $$\lim_{x\to 0}\lim_{y\to 0}\frac{xy(x^2-y^2)}{(x^2+y^2)}$$

$$=\lim_{x\to 0} 0 = 0.$$

For Simultaneous limit (double limit).

Let $\varepsilon > 0$,

Putting $x = r\cos\theta$, $y = r\sin\theta$

Then $$\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right|$$

$$=\left| r\cos\theta . r\sin\theta \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right|$$

$$=\left| r^2\sin\theta\cos\theta \frac{(\cos^2\theta - \sin^2\theta)}{1}\right|$$

$$= \left| r^2 \frac{\sin 2\theta}{2} \cdot \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} 2\sin 2\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2 \sin 4\theta}{4} \right|$$

$$\leq \frac{r^2}{4} = \frac{x^2+y^2}{4} < \varepsilon$$

if $\frac{x^2}{4} < \frac{\varepsilon}{2}$, $\frac{y^2}{4} < \frac{\varepsilon}{2}$

Or, if $x < \sqrt{2}\,\varepsilon$, $y < \sqrt{2}\,\varepsilon$

taking $\sqrt{2}\,\varepsilon = \delta > 0$. So that given $\varepsilon > 0$, there exists

$$\delta = \sqrt{2}\,\varepsilon > 0.$$

Such that

$$\left| xy \frac{(x^2-y^2)}{(x^2+y^2)} - 0 \right| < \varepsilon$$

Such that $|x-0| < \delta$, $|y-0| < \delta$

Thus we get

$$\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0.$$

Hence all the limits exist and are equal.

(3) Show that the double limit (Simultaneous limit)

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^3}{x^2+y^6}$$ does not exist.

Sol^n: Firstly let $(x,y)\to(0,0)$ through the line $y=x$ which is a line through the origin that is, putting $y=x$.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^3}{x^2+y^6} = \lim_{x\to 0} \frac{x\cdot x^3}{x^2+x^6}$$

$$= \lim_{x\to 0} \frac{x^2}{1+x^4} = 0.$$

Again let (x,y) approach $(0,0)$ through the curve $x=y^3$ that is putting $x=y^3$

Therefore $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{y^3y^3}{y^6+y^6}$$

$$=\frac{1}{2}.$$

Thus in the two approaches, the limits are different. Hence the simultaneous limit does not exist.

Sd.
Dr. Birendra
Asst. Prof. of Math.
Dept of Math.
D.K. College,
Dumraon