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Mathematics Hons & Sub.
B. Sc. Part-I
Paper-I
Topic: Assignment Problem.
(L.P)

Assignment Problem:

The name 'Assignment problem' Originates from the Classical problems where the objective is to assign a number of jobs to the equal number of persons at a minimum cost or maximum profit. To examine the nature of assignment problem, let there are n jobs to be performed and n-persons are available for doing these jobs, Assume that each person can do each job at a time, those with varying degree of efficiency. Let c_{ij} be the cost if the i^{th} person is assigned the j^{th} job, the problem is to find an assignment, i.e, which job should be assigned to which person. So that the total cost for performing all jobs is minimum.

Problem of this type are known as assignment problem.

Mathematical Formulation of Assignment Problem:

For the ballanced assignment problem. Let c_{ij} be the cost (Payment) incurred if the i^{th} person is assigned the j^{th} job. Then the assignment problem can be stated in the form of $n \times n$ matrix $[c_{ij}]$ of real numbers as given below:

Jobs

	1	2	3	---	---	n
1	c_{11}	c_{12}	c_{13}	--	c_{1j}	-- c_{1n}
2	c_{21}	c_{22}	c_{23}	--	c_{2j}	-- c_{2n}
3	c_{31}	c_{32}	c_{33}	--	c_{3j}	-- c_{3n}
	--	--	--	--	--	--
	--	--	--	--	--	--
n	--	--	--	--	--	--

If we define a variable x_{ij} which indicates the assignment of the i^{th} person to the j^{th} job.

i.e,

$$x_{ij} = \begin{cases} 1, \text{ if the } i^{th} \text{ person is assigned to the } j^{th} \text{ job.} \\ 0, \text{ if the } i^{th} \text{ person is not assigned to the } j^{th} \text{ job.} \end{cases}$$

Since for a particular one person, only one job can be assigned, we have

$$\sum_{j=1}^{n} x_{ij} = 1; \quad i = 1, 2, 3, --- n.$$

i.e, for a fixed i, only one $x_{ij} = 1$, and rest of $(n-1)$ x_{ij}'s are zeroes

Similarly,

$$\sum_{i=1}^{n} x_{ij} = 1; \quad j = 1, 2, 3 --- n.$$

The total assignment cost is given by

$$Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}.$$

Thus the assignment problem in the mathematical form is given as:

Minimise the total cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}; \text{ for } i = 1, 2, 3 --- n.$$

$$j = 1, 2, 3 --- n.$$

Subject to the Condition

(i) $\sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, 3, \ldots n$

and (ii) $\sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, 3 \ldots n$

$x_{ij} \geq 0$ with $x_{ij} = 0$ or 1.

Theorem / if in an assignment problem we add to or substract from every element of a row or a column of the cost matrix $[c_{ij}]$, then an assignment which minimises the total cost for the modified matrix also minimises the total cost for the original matrix i.e, an assignment which is optimum for the original one.

Proof :- Let $x_{ij} = x_{ij}^{*}$

Minimises $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} c_{ij}$ such that

$\sum_{i=1}^{n} x_{ij} = 1, \; \sum_{j=1}^{n} x_{ij} = 1$ for all $x_{ij} = 0$ or 1.

Then x_{ij}^* also minimise

$$z^* = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} c_{ij}^*$$

Where $c_{ij}^* = c_{ij} - u_i - v_j$ $(i = 1, 2, 3 \ldots n)$ and u_i and v_j are some real numbers, we have

$$z^* = \sum_{j=1}^{n}\sum_{j=1}^{n} x_{ij} c_{ij}^*$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}(c_{ij} - u_i - v_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} c_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} u_i - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} v_j$$

$$= z - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} v_j \sum_{i=1}^{n} x_{ij}$$

$$= z - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} v_j$$

Since $\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$ and $\sum_{i=1}^{n} u_i$ and $\sum_{j=1}^{n} v_j$ are independent of x_{ij}.

it follows that if z is minimum for $x_{ij} = x_{ij}^*$, then z^* must also be a minimum for x_{ij}^*.

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