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Mathematical Hons.

B. Sc. Part-III

Paper-V

Topic: Repeated limit
(Real Analysis)

Ques: Show that $\lim_{(x,y) \rightarrow (0,0)}$

$\frac{xy}{x^2+y^2}$ does not exist.

Solⁿ: - Let $f(x,y) = \frac{xy}{x^2+y^2}$

Putting $y = m_1 x$ where $x \rightarrow 0$

$$f(x,y) = \frac{m_1 x}{1+m_1^2}$$

$$\therefore \lim_{x \rightarrow 0} f(x,y) = \frac{m_1}{1+m_1^2}$$

if we put $y = m_2 x$, then

$$f(x,y) = \frac{m_2}{1+m_2^2}$$

$$\therefore \lim_{x \rightarrow 0} f(x,y) = \frac{m_2}{1+m_2^2}$$

Thus if we approach the point

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$(0,0)$ along a line with slope m_1 , the above limit is m_1
 While if we approach $(0,0)$ along a line with slope m_2 , the
 above limit is m_2

Therefore the Simultaneous
 limit $\frac{xy}{x^2+y^2}$ does not exist.

Ques! - Show that the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x=y=0. \end{cases}$$

Possesses no limit at the
 origin.

Coⁿ - Here $f(x,y) = \frac{xy}{x^2+y^2}$,
 if $x^2+y^2 \neq 0$,

and $f(0,0) = 0$.

if we approach the origin along
 any axis,

$$f(x,x) = 0.$$

if we approach $(0,0)$ along any

line $y = mx$ then,

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$$f(x, y) = f(x, mx) = \frac{x^2 \cdot mx}{x^4 + (mx)^2}$$

$$= \frac{mx}{x^2 + m^2}$$

$$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{mx}{x^2 + m^2} =$$

$$\frac{m \cdot 0}{0 + m^2} = 0$$

So, any line approach gives

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$$

But putting $y = mx^2$

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{x \cdot mx^2}{x^4 + (mx^2)^2} = \frac{m}{1 + m^2}$$

Which is different for the different m selected.

Hence $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Thus the function possesses no limit at the origin.

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Ques: Give an example to show that the order of iterated limits cannot be interchanged although the simultaneous limit does not exist.

Sol: Let us consider the funⁿ:

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

for iterated (repeated) and simultaneous limits at $(0, 0)$.

Let the variables approach the origin along the line $y = x$.

Putting $y = x$ in the function and letting $x \rightarrow 0$ we have

$$\lim_{x \rightarrow 0} \frac{xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

in this case $x \rightarrow 0$ along the line parallel to x -axis. Then let $y \rightarrow 0$ along y -axis, we have

$$\begin{aligned} \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy}{x^2 + y^2} &= \lim_{y \rightarrow 0} \frac{0 \cdot y}{0 + y^2} \\ &= \lim_{x \rightarrow 0} \frac{0}{y^2} = 0. \end{aligned}$$

Since the results obtained by these two methods of approach

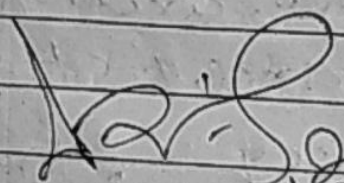
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we different, therefore the simultaneous limit does not exist.

for repeated limits,
we have $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{xy}{x^2+y^2} =$

$$\lim_{x \rightarrow 0} \frac{x \cdot 0}{x^2+0} = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0.$$

$$\text{and } \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{xy}{x^2+y^2} = \lim_{y \rightarrow 0} \frac{0 \cdot y}{0+y^2} \\ = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

Hence the order of the repeated limits can be interchanged.


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