Mathematics Hons.
B. Sc. Part - III
Paper - V
Topic: Repeated limits
(Real - Analysis)

Ques:- Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Sol^n:- Let $f(x,y) = \frac{xy}{x^2+y^2}$, $(x,y) \neq (0,0)$

Putting $y = m_1 x$ where $x \to 0$

$$f(x,y) = \frac{m_1}{1+m_1^2}$$

$$\therefore \lim f(x,y) = \frac{m_1}{1+m_1^2}$$

if we put $y = m_2 x$, then

$$f(x,y) = \frac{m_2}{1+m_2^2}$$

$$\therefore \lim f(x,y) = \frac{m_2}{1+m_2^2}.$$

Thus if we approach the point

$(0,0)$ along a line with slope m_1, the above limit is $\frac{m_1}{1+m_1^2}$ while if we approach $(0,0)$ along a line with slope m_2, the above limit is $\frac{m_2}{1+m_2^2}$

Therefore the simultaneous limit $\frac{xy}{x^2+y^2}$ does not exist.

Ques:- Show that the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x=y=0. \end{cases}$$

Possesses no limit at the Origin.

Sol^n:- Here $f(x,y) = \frac{x^2y}{x^4+y^2}$, if $x^2+y^2 \neq 0$,

and $f(0,0) = 0$.

if we approach the origin along any axis,

$$f(x,y) = 0.$$

if we approach $(0,0)$ along any line $y = mx$ then,

$$f(x,y) = f(x, mx^2) = \frac{x^2 \cdot mx^2}{x^4 + (mx^2)^2}$$

$$= \frac{m}{1+m^2}$$

1. $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx}{x^2+m^2} =$

$$\frac{m \cdot 0}{0+m^2} = 0$$

So, any line approach gives

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

But putting $y = mx^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x, mx^2)$$

$$= \frac{m}{1+m^2}$$

Which is different for the different m selected.

Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

Thus the function possesses no limit at the origin.

Ques: Give an example to show that the order of iterated limits cannot be interchanged although the simultaneous limit does not exist.

Sol:- Let us consider the funct:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

for iterated (repeated) and simultaneous limits at (0,0).
Let the variables approach the origin along the line $y=x$.
Putting $y=x$ in the function and letting $x \to 0$ we have

$$\lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

in this case $x \to 0$ along the line parallel to x-axis. Then let $y \to 0$ along y-axis, we have

$$\lim_{y\to 0} \lim_{x\to 0} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0.y}{0+y^2}$$

$$= \lim_{x\to 0} \frac{0}{y^2} = 0.$$

Since the results obtained by these two methods of approac are different, therefore the simultaneous limits does not exist.

For repeated limits,

we have $\lim\limits_{x\to 0} \lim\limits_{y\to 0} \frac{xy}{x^2+y^2} =$

$$\lim_{x\to 0} \frac{x.0}{x^2+0} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

and $\lim\limits_{y\to 0} \lim\limits_{x\to 0} \frac{xy}{x^2+y^2} = \lim\limits_{y\to 0} \frac{0.y}{0+y^2}$

$$= \lim_{y\to 0} \frac{0}{y^2} = 0.$$

Hence the order of the repeated limits can be interchanged.

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