Dated: 29/06/2021

Mathematics Hons.

B. Sc. Part - III

Paper - V^{th}

Topic: Continuity & Differenciability of functions. (Real Analysis)

Let $f(x,y)$ be a real valued function of two variables x and y defined on a set $S \subset R^2$ and $(a,b) \in S$. The function f is said to be continuous at (a,b) if for every $\varepsilon > 0$, there exist $\delta > 0$ such that

$|x-a| < \delta, \ |y-b| < \delta$

$\Rightarrow |f(x,y) - f(a,b)| < \varepsilon$

for all $(x,y) \in S$.

In other words $f(x,y)$ is said to be continuous at (a,b) if the simultaneous limit $\lim_{\substack{x \to a \\ y \to b}} f(x,y)$ exists and is equal to its functional value $f(a,b)$ at (a,b).

if f is continuous at (a,b). Then $f(x,b)$ is continuous at $x = a$ and $f(a,y)$ is continuous at $y = b$;

being functions of one variable. A function is said to be continuous in a region if it is continuous at every point of the same.

Algebra of Continuous functions:

if f and g be two real valued functions which are continuous at point $(a,b) \in S \subset R^2$; if λ, μ are real constants, then

(i) $\lambda f + \mu g$ is continuous at (a,b)

(ii) $|f|$ and g are continuous at (a,b)

(iii) (f/g) is continuous at (a,b) if $g(x,y) \neq 0 \ \forall (x,y) \in S$.

(iv) if $(a,b) > 0$, then $f(x,y) > 0$; $\forall (x,y) \in S$.

Partial Derivatives: Let $f(x,y)$ be a function of two variables x, y. The ordinary derivative of f with respect to one of the variables say x, keeping all other independent variables constant is called the partial derivative of $f(x,y)$ with respect to x and is denoted by

$$\frac{\partial f}{\partial x}, \quad \text{or } f_x \quad \text{or } f_x(x,y)$$

$$\therefore \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

While those with respect to y are denoted by

$$\frac{\partial f}{\partial y}, \quad \text{or } f_y \quad \text{or } f_y(x,y)$$

$$\therefore \frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Thus f is a function of two variables x and y, we have the partial derivatives of f with respect to x and y at a particular point $(a,b) \in$ domain f as denoted by

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} \text{ or } \frac{\partial f(a,b)}{\partial x} \text{ or } f_x(a,b)$$

and

$$\left(\frac{\partial f}{\partial y}\right)_{(a,b)} \text{ or } \frac{\partial f(a,b)}{\partial y} \text{ or } f_y(a,b).$$

Thus $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

and $f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$

in case the limit exist.

These are first partials.

The higher order partial derivatives are similarly

$$\frac{\partial}{\partial x}(f_x) = \frac{\partial^2 f}{\partial x^2} = f_{xx};$$

$$\frac{\partial}{\partial y}(f_x) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx};$$

$$\frac{\partial}{\partial x}(f_y) = f_{xy}; \quad \frac{\partial}{\partial y}(f_y) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

etc.

Thus

$$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_x(a,b)}{h}$$

and $f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_y(a,b)}{k}$.

Mean Value theorem for two variables.

Theorem! Let f be a real valued function of two variables x and y defined on $S \subset R^2$; if $f_x(x,y)$ exists for all $x, y \in S$ and $f_y(a,b)$ also exists. Then if point $(a+h, b+k) \in S$, then

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a+b) + \eta] \quad 0<\theta<1.$$

Where θ depends upon h and k and η is a function of k tending to zero with k, i.e,

$$\eta = \eta(k) \to 0 \text{ as } k \to 0.$$

Proof:- As $f_x(x,y)$ exists for all $(x,y) \in S$, $f_x(x, b+k)$ exists for all $x \in [a, a+h]$. Thus $f(x, b+k)$ is continuous on $[a, a+h]$ and differentiable $]a, a+h[$.

By Lagrange's Mean-value theorem, we have

$$f(a+h, b+k) - f(a, b+k) = h f_x(a+\theta h, b+k), \quad 0<\theta<1 \quad (1)$$

Also $f_y(a,b)$ exists, so that

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k} = f_y(a,b)$$

which implies $f(a, b+k) - f(a,b) = k[f_y(a,b) + \eta]$ — (2)

where η is a function of k and tends to zero as $k \to 0$. Substracting from (2) in (1) we get $f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta]$.

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