01.07.2021

Mathematics Hons.

B.Sc. Part - III

Paper - V^{th}.

Name Of the Topic: Continuity & differentiability of function.

Q:- Show that the function $f(x,y) = \frac{xy^3}{x^2+y^6}$, $x \neq 0$, $y \neq 0$ and $f(0,0) = 0$, is not continuous at $(0,0)$ in (x,y) together but the function is continuous in x alone and y alone at the origin.

Solution: Here $f(x,y) = \frac{xy^3}{x^2+y^6}$, $x \neq 0$, $y \neq 0$ and $f(0,0) = 0$.

Let $f(x,y)$ approaches $(0,0)$ through any line $y = mx$. we obtain,

$$\lim_{x \to 0} \frac{x \cdot m^3 x^3}{x^2 + m^6 x^6}$$

$$= \lim_{x \to 0} \frac{m^3 x^2}{1 + m^6 x^4}$$

$$= \frac{0}{1+0} = 0.$$

Again, let (x,y), approaches $(0,0)$ through the curve $x=y^3$. Then we have

$$\lim_{y\to 0} \frac{y^3 y^3}{y^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}.$$

Since the limit obtained by the two approaches are different, therefore the simultaneous limit of $f(x,y)$ at $(0,0)$ does not exist.

Hence the function is not continuous at $(0,0)$ in (x,y) together.

Putting either variable and then letting the other variable approach zero.

i.e, Putting $y=0$, $\lim_{x\to 0} \frac{x\cdot 0}{x^2+0}$

$$= \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Similarly Putting $x=0$,

$$\lim_{y\to 0} \frac{0\cdot y^3}{0+y^6} = 0.$$

$\therefore f(0,0) = 0.$

Therefore the function is continuous in x alone and in y alone at the origin.

Q: Construct an example of a function which is separately continuous but not continuous.

Sol^n: Let a function

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & \text{at } (0,0) \end{cases}$$

This function is not continuous at (0,0) for $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. We see by taking $(x,y) \to (0,0)$ through the curve $y = mx$, $m \neq 0$, we have

$$f(x,y) = \frac{x^2 \cdot m^2 x^2}{x^4 + m^4 x^4}$$

$$= \frac{m^2 x^4}{x^4(1+m^4)} = \frac{m^2}{1+m^4}.$$

which is not independent of m.

But $\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} \dfrac{x^2 \cdot 0}{x^4+0}$

$= 0 = f(0,0)$

and $\lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} \dfrac{0 \cdot y^2}{0+y^4}$

$= 0 = f(0,0)$

Thus f is separately continuous but it is not continuous at (0,0).

Q:- Investigate for continuity at $(2,3)$, the function f define by

$$f(x,y) = \begin{cases} 3xy, & (x,y) \neq (2,3) \\ 4, & (x,y) = (2,3) \end{cases}$$

Solution:

Here, given that the function $f(x,y) = 3xy$ when $(x,y) \neq (2,3)$

and $f(x,y) = 4$ when $(x,y) = (2,3)$

Now, $\lim_{(x,y)\to(2,3)} f(x,y) =$

$$\lim_{(x,y)\to(2,3)} 3xy = 3\cdot 2\cdot 3 = 18 \quad \text{--- (1)}$$

and $f(2,3) = 4$ --- (2)

from (1) and (2),

Since $\lim_{(x,y)\to(2,3)} (x,y) \neq f(2,3)$

Hence f is not continuous at $(2,3)$

if the function $f(x,y)$ had the value 18 at $(2,3)$, it would then be continuous at $(2,3)$.

Dr. Birendra Kr. (Retd.)
Asst. Prof. of Maths.
Dept. of D.K. College, Dumraon