Date / 30/06/2021

Mathematics Hons.

B. Sc. Part - III

Paper - V^{th}

Topic: Continuity and differentiability of a function.

A Sufficient Condition for Continuity at a point.

Theorem: Let f be a real valued function of x and y defined on $S \subset R^2$. A Sufficient Condition for f to be Continuous at (a, b) is that one of the partials say f_x (or f_y) exists and is bounded in S and the other partial f_y (or f_x) exists at (a, b).

Proof:- Let f_x exists and bounded in a neighbourhood of (a, b) and let $f_y(a, b)$ exist. Then for any point $(a+h, b+k)$ of this neighbourhood, we have by mean value theorem,

$$f(a+h, b+k) - f(a, b) = h f_x(a+\theta h, b+k) + k[f_y(a, b) + \eta] \quad (1)$$

where $0 < \theta < 1$ and $\eta \to 0$ as $k \to 0$

Since $f_x(a+\theta h, b+k)$ ~~f(a,b)~~ is bounded, letting $(h, k) \to (0, 0)$

(1) becomes

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$$

$\Rightarrow f$ is continuous at (a,b)

(c) Prove that a function of two variables continuous in a rectangle (boundary included) is uniformly continuous in the rectangle.

Soln:- A function $f(x,y)$ is said to be uniformly continuous over a rectangle R if for any ϵ, $\exists \delta > 0$ such that $|f(x_2,y_2) - f(x_1,y_1)| > \epsilon$ where (x_1,y_1) and (x_2,y_2) are any two points of R satisfying $|x_2 - x_1| \le \delta$ and $|y_2 - y_1| \le \delta$.

We now propose to prove the result that if $f(x,y)$ is continuous in the rectangle R (including boundary) then $f(x,y)$ is uniformly continuous in R. For this, we devide R into a finite number of sub-rectangles, s.t

$$|f(x_2,y_2) - f(x_1,y_1)| < \epsilon/2 \quad \text{--- (2)}$$

This is possible since f is continuous in R. Let δ be the positive number less than all

Sides of all sub-rectangles of R and let us choose any two points (x_1, y_1) and (x_2, y_2) of R satisfying

$$|x_2 - x_1| \leq \delta \text{ and } |y_2 - y_1| \leq \delta.$$

Thus (x_1, y_1) and (x_2, y_2) either belong to the same subrectangle or to two sub-rectangle which have a side or a part of a side in common. in the former case

$$|f(x_2, y_2) - f(x_1, y_1)| < \varepsilon/2 < \varepsilon \quad \text{--- (3)}$$

in the latter case, let (x_0, y_0) be a point on the common side of the two sub-rectangles. Then,

$$|f(x_2, y_2) - f(x_1, y_1)| \leq |f(x_2, y_2) - f(x_0, y_0)| + |f(x_0, y_0) - f(x_1, y_1)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = < \varepsilon \quad \text{--- (4)}$$

But (4) means that the function $f(x, y)$ is uniformly continuous over R. This completes the proof.

Q. Investigate the continuity at $(0,0)$ of $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

Solⁿ:- Here

$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, $(x, y) \neq (0,0)$ and $f(0,0) = 0$.

To test the continuity at the origin, we find that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \quad (1)$$

if we put $x = my$ in (1), we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{m^2y^2-y^2}{m^2y^2+y^2} = \lim_{y\to 0} \frac{m^2-1}{m^2+1}$$ which depends upon m.

Hence the function is not continuous at (0,0).

Q: Show that the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

Sol^n:- Let $x = r\cos\theta$ and

$y = r\sin\theta$.

$$\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{r\cos\theta . r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right|$$

$$= |r\cos\theta\sin\theta| \leq r = \sqrt{x^2+y^2} < \epsilon$$

if $x^2 < \frac{\epsilon^2}{2}$, $y^2 < \frac{\epsilon^2}{2}$

Or, if $|x| < \frac{\epsilon}{\sqrt{2}}$, $y < \frac{\epsilon}{\sqrt{2}}$

Thus $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon$, when

$|x| < \frac{\epsilon}{\sqrt{2}}, \; |y| < \frac{\epsilon}{\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

i.e, $\lim_{(x,y)\to(0)} f(x,y) = f(0,0)$

Hence f is continuous at $(0,0)$

Q. Show that $f(x,y) = e^{-1/x^2+y^2}$; $(x,y) \neq (0,0)$, $f(0,0) = 0$. is continuous.

Sol^n: The function given is

$f(x,y) = e^{-1/(x^2+y^2)}$; $(x,y) \neq (0,0)$

we calculate the first partials

$$f_x(x,y) = e^{-1/x^2+y^2} \cdot \left[\frac{2x}{(x^2+y^2)^2}\right],$$

$x \neq 0, y \neq 0, f_x(0,0) = 0.$

$$f_y(x,y) = e^{-1/(x^2+y^2)} \cdot \left[\frac{2x}{(x^2+y^2)^2}\right],$$

$x \neq 0, y \neq 0, f_y(0,0) = 0.$

Since both the first order partials are bounded, f is neccessarily continous every where.

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