Mathematics Hons.

B. Sc. Part - III

Paper - V^th

Topic: Differentiability of two variables.

Let f be a real valued function defined in a domain $D \subset R^2$ and let (a,b) be an interior point of D. f is said to be differentiable at the (a,b) if

$$\frac{f(a+h, b+k) - f(a,b) - (Ah+Bk)}{|h| + |k|} \to 0$$

as $h \to 0$ and $k \to 0$ ——— (1)

where A and B are independent of h and k.

In other words, f is said to be differentiable at (a,b), if for any $\varepsilon > 0$, there exist $\delta > 0$ such that

$$|f(a+h, b+k) - f(a,b) - (Ah+Bk)| \leq \varepsilon(|h| + |k|) \quad \text{——— (2)}$$

for $|h| < \delta$, $|k| < \delta$

where A and B are independent of h and k.

In this case $Ah+Bk$ is known as the differential of f at (a,b).

In the second definition if f is differentiable at (a,b), then by putting $k=0$ in (11), we get that f is partially differentiable with respect to the first variable at (a,b), i.e. $f_1(a,b)$ exists and similarly $f_2(a,b)$ exists.

In other form, (11) can be expressed as:

for any $\epsilon>0$, there exists $\delta>0$, s.t. $[f(a+h,b+k)-f(a,b)-f(a,b)h + f_2(a,b)k] < \epsilon(|h|+|k|)$, for $|h|<\delta, |k|<\delta$ (12)

In the equivalent form, the differentiability is defined as follows:

Let f be a function of two variables x, y defined in a certain neighbourhood N of (a,b). Then f is said to be differentiable at (a,b) if given $\epsilon>0$, there exists $\delta>0$ s.t.

$$|f(a+h,b+k)-f(a,b)-f_1(a,b)h + f_2(a,b)k| \le \epsilon\rho$$

where $\rho = |h|+|k|$, $(|h|<\delta, |k|<\delta)$

Or, $\rho = \sqrt{h^2+k^2}$

Or, $\rho = \max \rho \{|h|+|k|\}$

Uniform partial derivatives

A function f of the variables x, y is said to posses uniform partial derivative w.r.t. the first variable at the point (a,b) if for any $\epsilon > 0$, there exists $\delta > 0$ s.t

$$|f(a+h, b+k) - f(a, b+k) - Ah| \le \epsilon |h|$$

for $|k| \le |h| < \delta$

where A is independent of h and k.

Putting $k=0$, the value of the uniform partial derivative with respect to the first variable is equal to the partial derivative $f_1'(a,b)$. The uniform partial derivative is denoted by $f_1(a,b)$. Thus when $f_1(a,b)$ exists so that $f_1(a,b)$ and $f_1'(a,b) = A$.

Similarly $f_2'(a,b)$, the uniform partial derivative with respect to the second variable exists if

$$|f(a+h, b+k) - f(a+h, b) - Bk| \leq \varepsilon |k|$$

where $|h| \leq |k| < \delta$

obviously $f_2'(a,b) = f_2(a,b) = B$.

Strong uniform partial derivatives:

A function of two variables x, y is said to have strong uniform partial derivative w.r.t. the first variable if for any $\varepsilon > 0$, $\exists \delta > 0$, s.t

$$|f(a+h, b+k) - f(a, b+k) - f_1(a,b)h| \leq \varepsilon |h|$$

for $|h| < \delta$ $|k| < \delta$.

The value of strong uniform partial derivative $= f(a,b)$.

Similarly for the strong uniform partial derivative w.r.t. the second variable, we have

$$|f(a+h, b+k) - f(a+h, b) - f_2(a,b).k| < \varepsilon |k|$$

for $|h| < \delta$ $|k| < \delta$.

Dr. Birendra Kr. Singh
Asst. prof. of Math.
Dept. D.K. college
Dumraon