05.07.2021.

Mathematics Hons & Sub

B. Sc. Part - I.

Paper - I

Topic: Relation between the roots and co-efficients (Exp. based on the Theory of eq^n)

Q: Show that if the equation $x^3 - ax^2 + bx - c = 0$ has pair of roots of the form $\alpha(1 \pm i)$, where α is real, then $(a^2 - 2b)(b^2 - 2ca) = c^2$. Hence find the roots of the equation $x^3 - 7x^2 + 20x - 24 = 0$

Sol^n:- Let the third root of the given equation be β.

Now,

$$\Sigma\alpha = (\alpha + i\alpha) + (\alpha - i\alpha) + \beta$$
$$= 2\alpha + \beta = a \text{ --- (1)}$$

$$\Sigma\alpha\beta = (\alpha + i\alpha)(\alpha - i\alpha) + \beta(\alpha + i\alpha) + \beta(\alpha - i\alpha) = b$$

Or, $\alpha^2 + \alpha^2 + 2\alpha\beta = b$

Or, $2(\alpha^2 + \alpha\beta) = b$ --- (2)

Squaring (1) we get

$$4\alpha^2 + 4\alpha\beta + \beta^2 = a^2 \text{ --- (3)}$$

Multiplying (3) by 2 and Subtracting it from (3), we get

$$\beta^2 = \alpha^2 - 2\beta \quad (4)$$

Also as β is a root of the given equation, so we have

$\beta^3 - \alpha\beta^2 + b\beta - c = 0.$

Or, $\beta(\beta^2 + b) - \alpha\beta^2 = 0.$

Or, $\beta(a^2 - 2b) + b - a(a^2 - 2b) - c = 0$ from (4)

Or, $(a^2 - 2b)(b^2 - 2ac) = c^2$ proved

Now, we are to solve the eqn

$x^3 - 7x^2 + 20x - 24 = 0.$

Here $a = 7$, $b = 20$, $c = 24$

From (4), we get

$2\alpha + \beta = a$

Or, $2\alpha + 3 = 7$

Or, $\alpha = 2$

$\therefore$ The required roots are $\alpha(1 \pm i)$, β.

where $\alpha = 2$, $\beta = 3$

i.e, $2(1 \pm i)$, 3.

Q. Show that the condition, that the cubic equation $x^3 + px^2 + qx + r = 0$ should have two roots α, β connected by the relation $\alpha\beta + 1 = 0$ is $1 + q + pr + r^2 = 0$.

Solⁿ:- The given cubic equation

$$x^3 + px^2 + qx + r = 0$$

Let γ be the third roots of this equation,

then $\alpha\beta\gamma = -r$

Or, $(-1)\gamma = -r, \because \alpha\beta = -1$

$$\therefore \gamma = r.$$

But γ is a root of the given equation

$$\therefore \gamma^2 + p\gamma^2 + q\gamma + r = 0 \quad \text{--- (1)}$$

putting $\gamma = r$ in (1), we get

$$r^2 + pr^2 + qr + r = 0$$

Or, $r^2 + pr + q + 1 = 0$

which is the required condition.

Q:- Find the condition when the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ are connected by the relation $\alpha\beta = -1$, where $\alpha, \beta, \gamma, \delta$ are the roots of the given equation.

Sol^n:- Since $\alpha, \beta, \gamma, \delta$ are the roots of the given equation

$x^4 + px^3 + qx^2 + rx + s = 0$.

$\therefore \alpha\beta\gamma\delta = -s$

$\therefore \gamma\delta = -s, \quad \because \alpha\beta = -1$

Now, we can write

$x^4 + px^3 + qx^2 + rx + s$

$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= (x^2 - ax + \alpha\beta)(x^2 - bx + \gamma\delta)$

where $a = \alpha + \beta$, $b = \gamma + \delta$.

Or, $x^4 + px^3 + qx^2 + rx + s$

$= (x^2 - ax - 1)(x^2 - bx - s)$

$\therefore \alpha\beta = -1, \gamma\delta = -s$.

Comparing the co-efficients of the powers of x on both sides, we get

$p = -(a+b)$ ——— (1)

$q = -s - 1 + ab$ ——— (2)

and $r = b + as$ ——— (3)

Adding ① & ③
we get,

$$p+r = a(s-1)$$

Or $a = (p+r)(s-1)$

From ①, we get,

$$b = -a-p = -\frac{p+r}{s-1} - p$$

$$= -\frac{ps+r}{s-1}$$

Now substituting the value of a and b in ②, we get

$$q+s+1 = ab = \left(\frac{p+r}{s-1}\right)\left\{-\left(\frac{ps+r}{s-1}\right)\right\}$$

Or, $(q+s+1)(s-1)^2 = -(sp+r)(p+r)$

Or, $s^2+s^2(q-1)+s(p^2+pr-2q-1)+(1+q+pr+r^2)=0$

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