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Mathematics Hons.

B. Sc. Part - III

Paper - V^{th}

Topic: Necessary condition for Differentiability.

Theorem:- Let f be a real valued function of two variables defined in a certain neighbourhood N of a point (a,b). if f is differentiable on (a,b), then

(i) f is continuous at (a,b)

(ii) $f_x(a,b)$ and $f_y(a,b)$ both exist.

Proof:- We assume f to be differentiable on (a,b) then $f_x(a,b)$ and $f_y(a,b)$ exist.

Let $(a+h, b+k) \in N$. Then by definition of differentiability

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \text{--- (1)}$$

Where A and B are constant independent of h and k, and $\phi(h,k) \to 0$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$

$\therefore$ From (1), ~~$\psi(h,k) \to 0$ as $(h,k) \to (0,0)$~~

~~then,~~ we observe that

$[f(a+b, b+k) - f(a,b)] \to 0$ as $(h,k) \to (0,0)$.

implies f is continuous at (a,b)

For $k=0$ and $h \neq 0$, we have from (1),

$$\frac{f(a+h,b)-f(a,b)}{h} = A+\phi(h,0) \quad \text{--- (2)}$$

$\because \phi(h,0) \to 0$ as $h \to 0$.

$\therefore$ (2) provides that $f_x(a,b)$ exists and equal to A.

For $h=0$ and $k \neq 0$, we have from (1)

$$\frac{f(a,b+k)-f(a,b)}{k} = B+\psi(0,k)$$

$\because \psi(0,k) \to$ as $k \to 0$

$\therefore$ (3) Provides that $f_y(a,b)$ exists and equal to B.

Sufficient Condition for Differentiability.

Theorem: Let f be a real valued function of two variables defined in a certain neighbourhood N of a point (a,b) such that

(i) f_x is continuous at (a,b)
(ii) f_y exists at (a,b)
Then f_y is differentiable at (a,b)

Proof:- Let the point $(a+h, b+k)$ be any point of the neighbourhood N of (a,b) in which f_y exists then by mean value theorem for two variables,

$$f(a+h, b+k) - f(a,b) = hf_x(a+\theta h, b+k) + kf_y(a,b) + k\eta \quad \text{--- (1)}$$

for $0 < \theta < 1$, where $\eta = \eta(k) \to 0$ as $h \to 0$, $k \to 0$.

We write (1) as

$$f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\phi(h,k) + k\psi(h,k) \quad \text{--- (2)}$$

where $\phi(h,k) = f_x(a+\theta h, b+k) - f_x(a,b)$ and $\eta = \psi(h,k)$.

As f_x is continuous at (a,b),
$\phi(h,k) \to 0$ as $(h,k) \to (0,0)$
Also $\psi(h,k) = \eta \to 0$ as $(h,k) \to (0,0)$
Hence relation (2) ensures that f is differentiable at (a,b)

Theorem:- (1) Let $f_x(a,b)$ exists
(ii) $f_y(a,b)$ exists and $f_y(x,y)$ is continuous at (a,b)

then $f(x,y)$ is differentiable at (a,b).

Proof:- By Mean value theorem, we have

$$f(a+h, b+k) - f(a+h, b) = k f_y(a+h, b+\theta k) \quad \text{--- (1)}$$

for some θ between 0 and 1.

$\because f_y(x,y)$ is continuous at (a,b)

$$\therefore \lim_{x\to a,\, y\to b} f_y(x,y) = f_y(a,b)$$

Putting $x = a+h$, $y = b+\theta k$

$$\lim_{h\to 0,\, k\to 0} f_y(a+h, b+k) = f_y(a,b)$$

$$\therefore f_y(a+h, b+\theta k) = f_y(a,b) + \psi(h,k) \quad \text{--- (2)}$$

where $\lim_{\substack{h\to 0\\ k\to 0}} (h,k) \to 0$

From (1) and (2), we get

$$f(a+h, b+k) - f(a+h, b) = k f_y(a,b) + k\psi(h,k) \quad \text{--- (3)}$$

where $\lim_{h\to 0,\, k\to 0} \psi(h,k) = 0$

Now $\lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$

$$= f_x(a,b)$$

Therefore $f(a+h, b) - f(a,b) = h f_x(a,b) + h\phi(h,k)$

When $\lim\limits_{h\to 0, k\to 0} \phi(h,k) = 0$.

Now, we have

$f(a+h, b+k) - f(a,b) =$

$f(a+h, b+k) - f(a+h, b) + f(a+h, b) - f(a,b)$

$= h f_x(a,b) + k f_y(a,b) + h\phi(h,k) + k\psi(h,k)$

where $\lim\limits_{h\to 0, k\to 0} \phi(h,k) = \lim\limits_{h\to 0, k\to 0} \psi(h,k)$

Therefore the function $f(x,y)$ is differentiable at (a,b).

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