06.07.2021.

Mathematics Hons.

B. Sc. Part-III

Paper-V^{th}.

Topic: N & S Condition For differentiability.

Thm/. A function $f(x,y)$ of two variables is differentiable at a point (a,b) if (i) $f_x(a,b)$ exists (ii) $f_y(a,b)$ exists and $f_y(x,y)$ is continuous at (a,b).

Proof/. By Mean value theorem for two variables

$f(a+h, b+k) - f(a+h, b)$

$$= k f_y(a+h, b+\theta k) \quad \text{--- (1)}$$

For some θ between 0 and 1.

Since $f_x(x,y)$ is continuous at (a,b)

therefore,

$$\lim_{\substack{x\to 0 \\ y\to 0}} f_y(x,y) = f_y(a,b) \quad \text{--- (2)}$$

Now putting $x = a+h$, $y = b+\theta k$ in (2), we obtain

$$\lim_{\substack{h\to 0\\ k\to 0}} f_y(a+h,\, b+\theta k) = f_y(a,b) \quad \text{---} (3)$$

Therefore $f(a+h, b+\theta k)$

$$= f_y(a,b) + \psi(h,k)$$

where $\lim_{\substack{h\to 0\\ k\to 0}} \psi(h,k) = 0.$

From (1) and (3), we get

$f(a+h, b+k) - f(a+h, b)$

$$= k f_y(a,b) + k\psi(h,k) \quad \text{---} (4)$$

where $\lim_{\substack{h\to 0\\ k\to 0}} \psi(h,k) = 0.$

Now, $\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

$$= f_x(a,b)$$

Therefore

$$f(a+h, b) - f(a, b)$$

$$= h f_x(a, b) + h\phi(h, k)$$

when $\lim_{\substack{h \to 0 \\ k \to 0}} \phi(h, k) = 0.$

Again, we have

$$f(a+h, b+k) - f(a, b)$$

$$= f(a+h, b+k) - f(a+h, b)$$

$$+ f(a+h, b) - f(a, b)$$

$$= h f_x(a, b) + k f_y(a, b)$$

$$+ h\phi(h, k) + k\psi(h, k)$$

where $\lim_{\substack{h \to 0 \\ k \to 0}} f(h, k)$

$$= \lim_{h \to 0} \psi(h, k) = 0.$$

Hence the function $f(x,y)$ is differentiable at (a,b).

Examples

1. Show that the function f, where

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x=y=0 \end{cases}$$

is continuous, possesses partial derivatives but is not differentiable at the origin.

Solution: Let $\varepsilon > 0$ be given.
Let $x = r\cos\theta$, $y = r\sin\theta$

Then $f(r\cos\theta, r\sin\theta)$

$$= \frac{r\cos\theta \cdot r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= r\sin\theta\cos\theta$$

$$= \frac{1}{2} r\sin 2\theta < \varepsilon$$

if $x^2 < \frac{\varepsilon^2}{2}$, $y^2 < \frac{\varepsilon^2}{2}$

Or, if $|x| < \frac{\varepsilon}{2}$, $|y| < \frac{\varepsilon}{2}$

Thus $\left| \frac{xy}{x^2+y^2} - 0 \right| < \varepsilon$

when $|x| < \frac{\varepsilon}{\sqrt{2}}$, $|y| < \frac{\varepsilon}{\sqrt{2}}$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

$\lim_{(x,y)\to 0} f(x,y) = f(0,0)$

Hence $f(x,y)$ is continuous at the origin. Also

$f_x(0,0) = 0$ and $f_y(0,0) = 0$

if the function $f(x,y)$ is differentiable at the origin, then by definition

$df = f(h+k) - f(0,0) = Ah + Bk + h\phi + k\psi.$

where $A = f_x(0,0) = 0$

$B = f_y(0,0) = 0$

and ϕ, ψ tends to zero as $(h,k) \to (0,0)$

$$\therefore \frac{hk}{\sqrt{h^2+k^2}} = h\phi + k\psi.$$

Putting $k = mh$ and letting $h \to 0$ we get

$$\frac{m}{\sqrt{1+m^2}} = \lim_{h \to 0} (\phi + m\psi) = 0$$

which is not possible for arbitrary m.

Hence the function is not differentiable at the origin.

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