07.07.2021.

Mathematics Hons.
B. Sc. Part. I
Paper. I
Topic: Exp. based on roots and coefficient relation (Theory of Equation)

Solve $18x^4 + 9x^3 - 41x^2 + 24x - 4 = 0$ having given that the sum of two of the roots in unity.

Soln:- Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation.

Then

$$\alpha+\beta+\gamma+\delta = \frac{-9}{18} = \frac{-1}{2}$$

Also, given $\alpha+\beta = 1$

$$\therefore \gamma+\delta = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Let $\alpha\beta = h$ and $\gamma\delta = k$.

then α, β are the roots of $x^2 - x + h = 0$

and γ, δ are the roots of

$$x^2 - x\left(\frac{-3}{2}\right) + k = 0.$$

i.e, $2x^2 + 3x + 2k = 0$.

Hence, $9(x^2 - x + h)(2x^2 + 3x + 2k)$

$$= 18x^4 + 9x^3 - 42x^2 + 24x - 4$$

$$= 9\left[2x^4 + x^3 + x^2(2k - 3 + 2h) - x(2k - 3h) + 2hk\right]$$

Now, equating the coefficients of like powers of x, we get,

$$9(2k-3+2h) = -41 \quad \text{--- (1)}$$

$$-9(2k-3h) = 24 \quad \text{--- (2)}$$

and $$18hk = -4 \quad \text{--- (3)}$$

Adding (1) and (2), we get

$-27 + 45k = -17$

i.e $45h = 10$

$\therefore h = \frac{10}{45} = \frac{2}{9}$.

From (3), $hk = \frac{-2}{9}$

$\therefore k = \frac{-2}{9} \times \frac{2}{2} = -1$.

Hence, α, β are the roots of $x^2 - x + \frac{2}{9} = 0$

i.e, $9x^2 - 9x + 2 = 0$ --- (4)

and γ, δ are the roots of the equation

$$2x^2 + 3x - 2 = 0 \quad \text{--- (5)}$$

Solving (4), we get

$(9x^2 - 6x) - (3x - 2) = 0$

$\Rightarrow 3x(3x-2) - (3x-2) = 0$

$\Rightarrow (3x-2)(3x-1) = 0$.

$\therefore x = \frac{2}{3}, \frac{1}{3}$.

Again, solving (5), we get

$(2x^2+4x)-(x+2)=0.$

Or, $2x(x+2)-(x+2)=0.$

Or, $(x+2)(2x-1)=0.$

$\therefore x=-2, \frac{1}{2}.$

Hence the roots of the equation are $\frac{2}{3}, \frac{1}{3}, -2, \frac{1}{2}.$

Q. Find the condition that sum of the two roots of the equation $x^4+px^3+qx^2+rx+s=0$ be equal to the sum of the other two roots.

Sol^n:- Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation such that

$$\alpha+\beta=\gamma+\delta \quad —(1)$$

Also, sum to the roots

$$\alpha+\beta+\gamma+\delta=-p$$

$$\therefore \alpha+\beta=-\frac{p}{2}=\gamma+\delta \quad —(2)$$

and $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$

Date
Page

Or, $(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=q$

Or, $(-\frac{1}{2}p)(-\frac{1}{2}p)+\alpha\beta+\gamma\delta=q$

Or, $\alpha\beta+\gamma\delta=q-\frac{1}{4}p^2$ —— (3)

and $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r$

Or, $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-r$

Or, $\alpha\beta(-\frac{1}{2}p)+\gamma\delta(-\frac{1}{2}p)=-r$

Or, $(\alpha\beta+\gamma\delta)p=2r$ —— (4)

From (3) and (4), eliminating $\alpha\beta$ and $\gamma\delta$, we have the required condition

$$p(q-\frac{1}{4}p^2)=2r$$

Or, $p^3-4pq+8r=0$

Q: Find the condition that the roots of the equation $x^3+px^2+qx+r=0$ be in A.P.

Hence solve the equation

$$8x^3-12x^2-2x+3=0.$$

Solⁿ:- Let the roots be

$\alpha-\delta,\ \alpha,\ \alpha+\delta$

then,

$$\alpha-\delta+\alpha+\alpha+\delta=-p \quad \text{--- (1)}$$

$$(\alpha-\delta)\alpha+(\alpha-\delta)(\alpha+\delta)+\alpha(\alpha+\delta)=q \quad \text{--- (2)}$$

and $(\alpha-\delta)\alpha(\alpha+\delta)=-r$ --- (3)

From (1), we get

$$3\alpha=-p, \text{ or } \alpha=\frac{-p}{3} \quad \text{--- (4)}$$

$$\alpha^2-\alpha\delta+\alpha^2-\delta^2+\alpha^2+\alpha\delta=q$$

Or, $3\alpha^2-\delta^2=q$ --- (5)

and from (3),

$$\alpha(\alpha^2-\delta^2)=-r \quad \text{--- (6)}$$

Now,

we have to eliminate α and δ from (4), (5) and (6)

$\therefore \alpha=\frac{-p}{3}$, from (4) & (5)

$3\cdot\frac{p^2}{9}-\delta^2=q$, Or $p^2-3\cdot\delta^2=3q$

Or, $3\delta^2=p^2-3q$

$\therefore \delta^2=(p^2-3q)/3$

Now substituting values for α and δ in (6), we get

$$-\frac{p}{3}\left\{\frac{p^2}{9}-\frac{p^2-3q}{3}\right\}=-r$$

Or, $\frac{p}{3}\left\{\frac{p^2-3p^2+9q}{9}\right\}=r$

Or, $p(-2p^2+9q)=27r$

Or, $-2p^3+9pq=27r$

$\therefore 2p^3-9pq+27r=0.$

Which is the required condition

Now, we are to solve the given equation $8x^3-12x^2-2x+3=0$

Here, $3\alpha=\frac{12}{8}, \therefore \alpha=\frac{1}{2}$

$3\alpha^2-\delta^2=\frac{-2}{8}$ Or, $3\frac{1}{4}-\delta^2=\frac{-1}{4}$

Or, $\delta^2=\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1, \therefore \delta=\pm 1$

~~Hence the roots are~~

$\alpha-\delta, \alpha, \alpha+\delta$ i.e, $\frac{1}{2}+1, \frac{1}{2}, \frac{1}{2}-1.$

i.e, $\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}$.

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