07.07.2021.

Mathematics Hons.

B.Sc. Part-III

Paper-Vth.

Topic: Continuity and differentiability of a function.

Question: Show that the function

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x^2+y^2)\neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$$

is differentiable at the origin.

Solution:- At the point origin

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0.$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,0+K)-f(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{0}{K} = 0.$$

$$\therefore f_x(0,0) = 0 = f_y(0,0)$$

Also when $x^2+y^2 \neq 0$

$$f_x(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

Putting $x = r\cos\theta$
$y = r\sin\theta$

$f_x(x,y) = |r|\,|\sin\theta|\,|\cos^2\theta - \sin^2\theta - 4\sin^2\theta\cos^2\theta|$

$\leq 6|r| = 6\sqrt{x^2+y^2}$

$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y) \leq 6 \cdot \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

Thus $\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0$

Hence f_x is continuous at $(0,0)$ and f_y exists.
Since these conditions are sufficient for differentiability at $(0,0)$, therefore the function is differentiable at the origin.

Q:- Investigate the partial derivatives and the uniform partial derivatives at $(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^{1/2}} & \text{where } (x,y) \neq (0,0) \\ 0, & \text{where } (x,y) = (0,0) \end{cases}$$

Solution: Here the function is

$$f(x,y) = \frac{xy}{(x^2+y^2)^{1/2}}, \text{ when } (x,y) \neq (0,0)$$

and $f(x,y) = 0$ when $(x,y) = (0,0)$

At the point origin

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0}{k} = 0.$$

Now, for uniform partial derivatives $f_x'(0,0)$ to exists, we have

$$\lim_{h\to 0, k\to 0} \frac{f(h,k) - f(0,k) - f_x(0,0)h}{h}$$

$$= 0.$$

$$\text{Or, } \lim_{h\to 0, k\to 0} \frac{hk}{(h^2+k^2)^{1/2}} = 0.$$

By putting $|k| = \frac{1}{2}|h|$, we get

$$\left|\frac{k}{(h^2+k^2)^{1/2}}\right| = \frac{\frac{1}{2}|h|}{|h|\left(1+\frac{1}{4}\right)^{1/2}}$$

$$= \frac{\frac{1}{2}}{\left(1+\frac{1}{4}\right)^{1/2}}$$

$\therefore \lim\limits_{\substack{h\to 0,\, k\to 0 \\ |k|\le|h|}} \frac{k}{(h^2+k^2)^{1/2}}$ doesnot exi[st]

$\therefore f'_x(0,0)$ doesnot exist.

Similarly, we can show that $f'_y(0,0)$ doesnot exist.

Thus the uniform partial derivatives at $(0,0)$ donot exist for the given function $f(x,y)$.

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