05.07.2021

Mathematics Hons & Sub.
B. Sc. Part-I.
Paper-I

Topic: Symmetric function of the roots.

Q:- if α, β, γ be the roots of $x^3 - px^2 + qx - r = 0$. find the value of $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

Sol:- Let α, β, γ be the roots of the equation

$$x^3 - px^2 + qx - r = 0$$

$\therefore \alpha + \beta + \gamma = p$, i.e $\sum\alpha = p$ — (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = q$, i.e $\sum\alpha\beta = q$ — (2)

and $\alpha\beta\gamma = r$ — (3)

Now, $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$

$$= \sum\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)$$

$$= \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\alpha^2+\gamma^2}{\alpha\gamma}$$

$$= \frac{\gamma(\alpha^2+\beta^2) + \alpha(\beta^2+\gamma^2) + \beta(\alpha^2+\gamma^2)}{\alpha\beta\gamma}$$

$$= \frac{\sum\alpha^2\beta}{\alpha\beta\gamma} \quad \text{— (4)}$$

Since $\Sigma\alpha\Sigma\alpha\beta =$

$$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= \Sigma\alpha^2\beta + 3\alpha\beta$$

$$\therefore \Sigma\alpha^2\beta = \Sigma\alpha\Sigma\alpha\beta - 3\alpha\beta\gamma$$
$$= (-p)q - 3r \quad [\text{from } (1), (2) \text{ and } (3)]$$
$$= pq - 3r$$

Hence from (4), we get

$$\Sigma\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) = \frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma} = \frac{pq-3r}{r}$$
$$= \frac{pq}{r} - 3.$$

Q:- Find the value of $\Sigma\frac{\alpha\beta}{\gamma^2}$ for the equation $x^4+px^3+qx^2+rx+S=0$ where $\alpha, \beta, \gamma, \delta$ are its roots.

Solⁿ:- If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4+px^3+qx^2+rx+S=0$, then $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\Sigma\alpha\beta\gamma = -r$ and $\alpha\beta\gamma\delta = S$.

Now,

Since we know that

$$\Sigma\alpha\beta \cdot \Sigma\frac{1}{\alpha^2} = (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta).$$
$$\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)$$

$$= \sum \frac{\alpha}{\beta} + \sum \frac{\alpha\beta}{\gamma^2} \quad \text{--- (1)}$$

Again, $\sum \alpha \sum \frac{1}{\alpha} =$

$$(\alpha+\beta+\gamma+\delta)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right)$$

$$= 4 + \sum\left(\frac{1}{\beta}\right)$$

Or, $\sum\left(\frac{\alpha}{\beta}\right) = \sum \alpha \sum \frac{1}{\alpha} - 4$

$$= (-p)\left[\frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}\right] - 4$$

$$= (-p)\left(\frac{-r}{s}\right) - 4 = \frac{pr}{s} - 4 \quad \text{--- (2)}$$

And $\sum \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$

$$= \left(\sum \frac{1}{\alpha}\right)^2 - 2\sum \frac{1}{\alpha\beta}$$

$$= \left[\frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta}\right]^2 - 2\left[\frac{\sum \alpha\beta}{\alpha\beta\gamma\delta}\right]$$

Or, $\sum \frac{1}{\alpha^2} = \left(\frac{-r}{s}\right)^2 - 2\left(\frac{q}{s}\right)$

$$= \frac{r^2 - 2qs}{s^2} \quad \text{--- (3)}$$

From (1), (2) and (3), we have

(q) $\left(\frac{r^2-2qs}{s^2}\right) = \left(\frac{pr}{s}-4\right) + \Sigma \frac{\alpha\beta}{\gamma^2}$

Or, $\Sigma \frac{\alpha\beta}{\gamma^2} = \frac{qr^2 - 2qrs - prs + 4s^2}{s^2}$.

Q: Find the value of symmetric function $\Sigma (\alpha-\beta)^2/(\alpha+\beta)^2$, where α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$.

Soln:- Here α, β, γ are the roots of the equation

$x^3+px^2+qx+r=0$.

$\therefore \alpha+\beta+\gamma = -p$ ——— (1)

$\alpha\beta+\beta\gamma+\gamma\alpha = q$ ——— (2)

and $\alpha\beta\gamma = -r$ ——— (3)

Now, $\Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 =$

$\Sigma\left[\frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2}\right]$

$= \Sigma\left[1 - \frac{4\alpha\beta}{(\alpha+\beta)^2}\right]$

$= 3 - 4\Sigma\frac{\alpha\beta}{(\alpha+\beta)^2} = 3 - 4\left[\frac{\Sigma\alpha\beta(\beta+\gamma)^2(\gamma+\alpha)^2}{(\alpha+\beta)^2(\beta+\gamma)^2(\gamma+\alpha)^2}\right]$

But $\Sigma\alpha\beta(\beta+\gamma)^2(\gamma+\alpha)^2$

$= \Sigma\alpha\beta(\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha)^2$

$$= \Sigma\alpha\beta(\gamma^2+q)^2, \quad (\because \Sigma\alpha\beta = q)$$

$$= \Sigma\alpha\beta(\gamma^4+2\gamma^2 q+q^2)$$

$$= \Sigma\alpha\beta\gamma^4 + 2q\Sigma\alpha\beta\gamma^2 + q^2\Sigma\alpha\beta$$

$$= \alpha\beta\gamma\Sigma\alpha^2 + 2q\alpha\beta\gamma\Sigma\alpha + q^2\Sigma\alpha\beta$$

$$= (-r)(3pq-p^3-3r) + 2q(-r)(-p) + q^2(q)$$

$$= p^3r - pqr + 3r^2 + q^3$$

And $(\alpha+\beta)^2(\beta+\gamma)^2+(\gamma+\alpha)^2$

$$= (r-pq)^2$$

Thus $\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2 = 3 - \dfrac{4p^2r-pqr+3r^2+q^2}{(r-pq)^2}$

$$= \frac{3p^2q^2 - 2pqr - 4p^3 - 9r^2 - 4q^3}{(r-pq)^2}$$

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