09.07.2021

Mathematics Hons.

B. Sc. Part - III

Paper - V^{th}

Topic: Schwarz's theorem

Statement: If (a,b) be a point in the domain of definition of a function $f(x,y)$ such that

(i) $f_x(x,y)$ and $f_y(x,y)$ exist in a certain neighbourhood of (a,b)

(ii) One of f_{xy} and f_{yx} is continuous at (a,b)

then the other exists at (a,b) and $f_{xy}(a,b) = f_{yx}(a,b)$.

Proof: The given condition means that there exists a certain neighbourhood of (a,b) at every point (x,y) of which $f_x(x,y)$, $f_y(x,y)$ and $f_{xy}(x,y)$ exist.

Let $\phi(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$

$g(y) = f(a+h, y) - f(a,b)$

Therefore $\phi(h,k) = g(b+k) - g(b)$ — (1)

We have f_y exists in a neighbourhood of (a,b), the function $g(y)$ is derivable in $[b, b+k]$.

Therefore from (1)

$\phi(h,k) = kf'(b+\theta k)$ $(0<\theta<1)$

by applying mean value thm

$= k\,[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)]$ (2)

Also we have f_{xy} exists in a neighbourhood of (a,b), the function $f_y(x, b+\theta k)$ of x is derivable w.r. to x in $(a, a+h)$ and therefore by applying the Mean value theorem in (2) we have

$\phi(h,k) = hk\,f_{xy}(a+\theta' h, b+\theta k)$,

$0<\theta'<1$.

$$\Rightarrow \frac{1}{k}\left[\frac{f(a+h, b+\theta k) - f(a, b+k)}{h} - \frac{f(a+h, b) - f(a,b)}{h}\right]$$

$$= f_{xy}(a+\theta' h, b+\theta k)$$

Also we have by the condition (given) $f_x(x,y)$ exists in a neighbourhood of (a,b), there fore as $h \to 0$ we get

$$\frac{f_x(a, b+k) - f_x(a,b)}{h} = \lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k)$$

Letting $k \to 0$, since $f_{xy}(x,y)$ is continuous at (a,b) we get

$$f_{yx}(a,b) = \lim_{k\to 0} \lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k)$$

$$= f_{xy}(a,b).$$

Young's theorem:

Statement: if f_x and f_y are both differentiable at a point (a,b) of the domain of definition of a function f, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Proof: Let $(a+h, b+h)$ be a point of neighbourhood N of (a,b).

Suppose that

$\phi(h,h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a,b)$

$G(x) = f(x, b+h) - f(x,b)$

$\therefore \phi(h,h) = G(a+h) - G(a)$ —— (1)

$\because f_x$ exists in a neighbourhood of (a,b), the function $G(x)$ is differentiable in $]a, a+h[$.

$\therefore$ By Lagrange's Mean value theorem, we have from (1)

$\phi(h,h) = hG'(a+\theta h)$, for $0<\theta<1$

$= h \{ f_x(a+\theta h, b+h) - f_x(a+\theta h, b) \}$

Again, since f_x is differentiable at (a,b), we have

$f_x(a+\theta h, b+h) - f_x(a,b) = \theta h f_{xx}(a,b) + h f_{yx}(a,b) + \theta h \phi_1(h,h) + h \psi_1(h,h)$ — (3)

and $f_x(a+\theta h, b) - f_x(a,b) = \theta h f_{xx}(a,b) + \theta h \phi_2(h,h)$ — (4)

where ϕ_1, ψ_1, ϕ_2 all tend to zero as $h \to 0$

Now, from (2), (3) and (4), we have

$$\frac{\phi(h,h)}{h^2} = f_{yx}(a,b) + \theta \phi_1(h,h) + \psi_1(h,h) - \theta \phi_2(h,h) \quad \text{— (5)}$$

Similarly considering

$H(y) = f(a+h, y) - f(a, y)$

we can show that

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \phi_3(h,h) + \theta' \psi_2(h,h) - \theta' \psi_3(h,h) \quad \text{— (6)}$$

where ϕ_3, ψ_2, ψ_3 all tend to zero as $h \to 0$

Now, taking the limit as $h \to 0$, we get from (5) and (6)

$$\lim_{h \to 0} \frac{\phi(h,k)}{h^2} = f_{xy}(a,b) = f_{xy}(a,b)$$

Example:

(1) if $f(x,y) = \begin{cases} (x^2+y^2)\log(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that f_{xy} and f_{yx} are not continuous at $(0,0)$ but $f_{xy}(0,0) = f_{yx}(0,0)$. Can you account for the equality?

Soln.:

Here, $f(x,y) = (x^2+y^2)\log(x^2+y^2)$, $(x,y) \neq (0,0)$

and $f(0,0) = 0$.

For $(x,y) \neq (0,0)$, we have

$f_x = 2x\log(x^2+y^2) + (x^2+y^2)\cdot\frac{1}{(x^2+y^2)}\cdot 2x$

$= 2x\{\log(x^2+y^2)+1\}$

Similarly, $f_y = 2y\{\log(x^2+y^2)+1\}$

Now $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \frac{h^2\log h^2}{h} = \lim_{h\to 0} 2h\log h = 0$

Similarly, $f_y(0,0) = 0$

Now, $f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$

$= \lim_{k\to 0} \frac{0}{k} = 0.$

and Similarly $f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$

$= \lim_{h\to 0} \frac{0}{h} = 0.$

Hence $f_{xy}(0,0) = f_{yx}(0,0) = 0.$

Sd.
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