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Mathematics Hons. & Sub.
B.Sc. Part-I.
Paper-I
Topic: Symmetric function of the roots.

Symmetric function: An expression of the roots of an equation is called a symmetric function of the roots of the given equation, which remain unchanged, when any two of the roots are interchanged.

Example: $\Sigma \alpha\beta$ represent $\alpha\beta + \beta\gamma + \gamma\alpha$ is a symmetric function of the roots, α, β, γ of a cubic equation.

Order of a symmetric function: The highest degree in which any root occurs in the function is called the order of a symmetric function of the roots of an equation.

Example: The order of the symmetric function $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta = 2$.

Weight of the Symmetric function

The sum of degrees of all the roots occuring in any term of the function is called the weight of the symmetric function.

Example: The weight of the symmetric function

$\Sigma \alpha\beta^2\gamma^4$ is $1+2+4=7$.

~~Examples~~ (1) If α, β, γ be the roots of the cubic equation $x^3+px^2+qx+r=0$, calculate the value of

(i) $\Sigma\alpha^3$ (ii) $\Sigma \frac{\beta^2+\gamma^2}{\beta+\gamma}$

Soln: Here, $\alpha+\beta+\gamma = -p$

$\alpha\beta+\beta\gamma+\gamma\alpha = q$

$\alpha\beta\gamma = -r$

(i) We know that

$\Sigma\alpha\,\Sigma\alpha^2 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2)$

$= \Sigma\alpha^3 + \Sigma\alpha^2\beta$

where $\Sigma\alpha^2\beta = (3r-pq)$

$\therefore \Sigma\alpha^3 = \Sigma\alpha\,\Sigma\alpha^2 - \Sigma\alpha^2\beta$

$= (-p)(p^2-2q) - (3r-pq)$

$= -p^3+2pq-3r+pq$

$= 3pq-3r-p^3.$

(i) $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{\beta^2+\gamma^2}{\beta+\gamma} + \frac{\gamma^2+\alpha^2}{\gamma+\alpha} + \frac{\alpha^2+\beta^2}{\alpha+\beta}$

$= \frac{\sum(\gamma+\alpha)(\alpha+\beta)(\beta^2+\gamma^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}$

Now,

$\sum(\gamma+\alpha)(\alpha+\beta)(\beta^2+\gamma^2)$

$= \sum(\beta^2+\gamma^2)\{\alpha^2+(\alpha\gamma+\alpha\beta+\beta\gamma)\}$

$= \sum(\beta^2+\gamma^2)(\alpha^2+q)$

$= \sum\alpha^2(\beta^2+\gamma^2) + q\sum(\beta^2+\gamma^2)$

$= 2\sum\alpha^2\beta^2 + 2q\sum\alpha^2$

$= 2(q^2-2pr) + 2q(p^2-2q)$

$= 2q^2 - 4pr + 2p^2q - 4q^2$

$= -2q^2 - 4pr + 2p^2q$

$= 2p^2q - 4pr - 2q^2$

And $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$= (\beta+\gamma)(\gamma\alpha+\beta\gamma+\alpha^2+\alpha\beta)$

$= \sum\alpha^2\beta + 2\alpha\beta\gamma$

$= 3r - pq + 2(-r)$

$= 3r - pq - 2r$

$= r - pq$

Therefore $\sum\frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{2p^2q-4pr-2q^2}{r-pq}$

(2) if α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ then find the value of

(i) $\Sigma(\alpha\beta)^2$ (ii) $\Sigma\alpha^2$ (iii) $\Sigma\alpha^2\beta$

(iv) $\Sigma\alpha^3\beta^3$ (v) $\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma}$ (vi) $\Sigma\frac{1}{\alpha}$.

Soln:- (i) Here α, β, γ are the roots of the equation

$$x^3 + px^2 + qx + r = 0.$$

Therefore $\alpha + \beta + \gamma = -p$

i.e $\Sigma\alpha = -p$ — (1)

$\alpha\beta + \beta\gamma + \alpha\gamma = q$

i.e, $\Sigma\alpha\beta = q$. — (2)

and $\alpha\beta\gamma = -r$ — (3)

$\therefore (\Sigma\alpha\beta)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$

$= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$

$= \Sigma\alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$\therefore \Sigma(\alpha\beta)^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha$

$= (q)^2 - 2(-r)(-p)$

$= q^2 - 2pr$

Thus $\Sigma(\alpha\beta)^2 = q^2 - 2pr$

(ii) $(\Sigma\alpha)^2 = (\alpha + \beta + \gamma)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$

$\therefore \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-p)^2 - 2q = p^2 - 2q$

(iii) We know that

$$\Sigma\alpha \cdot \Sigma\alpha\beta = (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= \Sigma\alpha^2\beta + 3\alpha\beta\gamma$$

Or, $(-p)(q) = \Sigma\alpha^2\beta + 3(-r)$

$$\therefore \Sigma\alpha^2\beta = 3r - pq.$$

(iv) We know that

$$\Sigma\alpha\beta\,\Sigma\alpha^2\beta^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$$
$$= \Sigma\alpha^3\beta^3 + \Sigma\alpha^3\beta^2\gamma$$
$$= \Sigma\alpha^3\beta^3 + \alpha\beta\gamma\,\Sigma\alpha^2\beta$$
$$\therefore \Sigma\alpha^3\beta^3 = \Sigma\alpha\beta\,\Sigma\alpha^2\beta^2 - \alpha\beta\gamma\,\Sigma\alpha^2\beta$$
$$= q(q^2-2pr) - (-r)(3r-pq)$$
$$= q^3 - 2pqr + 3r^2 - pqr$$
$$= q^3 - 3pqr + 3r^2$$

(v) $$\Sigma\frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$$
$$= \frac{\alpha(\beta^2+\gamma^2)+\beta(\gamma^2+\alpha^2)+\gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$$
$$= \frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma}, \quad \because \Sigma\alpha^2\beta = 3r-pq$$
$$\therefore \Sigma\frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{3r-pq}{-r} = \frac{pq-3r}{r}$$

(vi) $$\Sigma\frac{1}{\alpha} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$$
$$= \frac{q}{-r} \quad \therefore \Sigma\frac{1}{\alpha} = -\frac{q}{r}$$

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