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Dept. of Mathematics.
B.Sc. Part II
(Maths. Hons.)
Paper-III

Real Analysis: (Dedkind's theory)

Theorem:- if α and β are two cuts then prove that $\alpha\beta = \beta\alpha$.

Proof:- Let $\alpha = (A_1, A_2)$ and $\beta = (B_1, B_2)$ be any two cuts. then to prove that

$$(A_1, A_2) \cdot (B_1, B_2) = (B_1, B_2) \cdot (A_1, A_2)$$

i.e, $\alpha\beta = \beta\alpha$.

Let $(U_1, U_2) = (A_1, A_2)(B_1, B_2)$

and $(V_1, V_2) = (B_1, B_2) \cdot (A_1, A_2)$

Let $u_1 \in U_1$ and $v_1 \in V_1$

Then u_1 is of the form $x_1 y_1$ where $x_1 \in A_1$ and $y_1 \in B_1$, and v_1 is of the form $y_1 x_1$ where $x_1 \in A_1, y_1 \in B_1$. Since the commutative law holds for rational numbers. So we have

$$x_1 y_1 = y_1 x_1$$

which shows that $U_1 = V_1$ and consequently $(U_1, U_2) = (V_1, V_2)$

Thus $(A_1, A_2) \cdot (B_1, B_2) = (B_1, B_2)(A_1, A_2)$

Theorem:- State and prove Dedkind's Theorem.

Statement:- Let R_1 and $R_2 \subset R$ where R is the set of real number such that

(i) $R_1 \neq \phi$, $R_2 \neq \phi$ i.e. each class exist

(ii) $R_1 \cup R_2 = R$

(iii) if $x \in R_1$, $y \in R_1 \Rightarrow x < y$

i.e. every member of R_1 is less than every member of R_2.

Then $\exists$ a greatest number of R_1 or a least number of R_2.

Proof:- Here we shall show that only two types of section are possible.

(i) either R_1 has a greatest member

or

(ii) R_2 has a least member.

Let A_1 and A_2 are the set of rational (real) of R_1 and R_2 respectively. Consider the section (A_1, A_2) with $A_1 \subset R_1$ and $A_2 \subset R_2$.

If x and y are rational numbers, then $x \in A_1 \Rightarrow x \in R_1$ and $y \in A_2$ and conversely since x and y are rational numbers.

We find that $x \in R_1 \Rightarrow x \in A_1$ (3)
and $y \in R_2 \Rightarrow y \in A_2$.
Here following three cases arises:
CASE I :- The lower class A_1 has greatest member (say l) and the upper class A_2 has no least member.
Since A_1 is a proper subset of A_1. So $l \in A_1 \Rightarrow l \in R_1$. Let if possible l is not the greatest member of A_1 but l' is the greatest member of A_1. l' may be a rational. Then $l' > l$.
But R is a dense set and so there exist an infinite number of rationals between l and l'. These rational numbers are less than l' and so, they will belongs to A_1. Hence there exist an infinite number of rationals in A_1 which are greater than l. Thus l is not the greatest member of A_1. This contradiction shows that l is the greatest member of R_1.
$\therefore$ l is the greatest member of $A_1 \Rightarrow l$ is the greatest member of R_1.

CASE II :- A_1 has no greatest member and A_2 has a least member say m.
Since as above, we can say that R_1 has no greatest member and R_2 has a least member m.

CASE III - A_1 has no greatest member and A_2 has no least member.

From above discussion, we find that there are only two types of sections of real numbers:

(i) R_1 has a greatest member but R_2 has no least.

(ii) R_1 has no greatest member but R_2 has a least.

Hence in either case R_1 has a greatest member or R_2 has a least. Third possibility namely R_1 has no greatest member and R_2 has no least member does not find a place in the section of real numbers.

This completes the result.