Date - 13/01/22.

Dept. of Maths.
B.Sc. Part I
(Maths. Hons)
Paper - I

Name of the Topic:

Theory Of Equation.

Q. State and prove fundamental theorem Of Algebra.

Statement:- Every polynomial equation with real coefficients has at least one root

Or,

Every equation of n^{th} degree has n roots and no more.

Proof:- Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ —(1)

be a general equation of n^{th} degree. Then we have to prove that their equation has n roots.

In the first stage of the proof, let us assume that every equation has a root. This is the subject of Complex analysis. So, it is taken for granted here.

Let α_1 be one of the roots of (2) the equation. Then it is clear that $(x-\alpha_1)$ is a factor of $f(x)$. it means that when the division of $f(x)$ is done by $(x-\alpha_1)$ then there is no remainder.

Let $\phi_1(x)$ be the quotient in this case. Thus we can write

$$f(x) = (x-\alpha_1)\phi(x) \quad \text{———} \quad (1)$$

it is to be noted here that $\phi_1(x)$ is a polynomial of $(n-1)^{th}$ degree. Then according to our elementary assumption this $\phi_1(x)$ will also have a root and let, α_2 be its root

then again $(x-\alpha_2)$ will be a factor of $\phi_1(x)$ and we can write $\phi_1(x) = (x-\alpha_2)\phi_2(x)$ ——— (2)

from (1) & (2), we have

$$f(x) = (x-\alpha_1)(x-\alpha_2)\phi_2(x). \quad \text{———} \quad (3)$$

it may be noted here that $\phi_2(x)$ is a polynomial of $(n-2)^{th}$ degree.

Again $\phi_2(x)$ has a root. Let α_3 be its root. Then $(x-\alpha_3)$ is a factor of $\phi_2(x)$

$$\phi_2(x) = (x-\alpha_3)\phi_2(x) \quad \text{———} \quad (4)$$

from (3) & (4) we have

$$f(a) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\phi_3(x) \quad \text{———} \quad (5)$$

when $\phi_3(x)$ is a polynomial of $(n-3)^{th}$ degree

Proceeding in this way, we observe that after n^{th} factor degree of the quotient polynomial becomes zero and so it cannot. Then we can write

$f(x) = (x-\alpha_1)(x-\alpha_2) - - - (x-\alpha_n)k$ — (B)

Now, equating the coefficient of x^n from (A) and (B) we have $k = a_0$.

Thus $f(x) = a_0(x-\alpha_1)(x-\alpha_2) - - - (x-\alpha_n)$

if we have put $x = \alpha_1, \alpha_2, - - - \alpha_n$ successively in the above equation is satisfied. Thus there are n^{th} roots of the given equation.

It is also clear that no other quantity than these n can satisfy the above equation having more than n roots doesn't arise.

Thus we conclude that every eqn of n^{th} degree has n roots and no more.

Q. (2) Establish Cardon's method of solving cubic equation.

Sol^n:

The general equation of a cubic

$$ax^3 + 3bx^2 + 3cx + d = 0$$

In order to obtain the solution of a cubic, we first reduce it to the cubic in which the second degree term is absent. for this, we adopt the substitution

$$z = ax + b.$$

$ax = z - b, \therefore x = \frac{z-b}{a}$

Substituting this value of x on a general equation of cubic, we have.

$$a\left(\frac{z-b^3}{a}\right) + 3b\left(\frac{z-b^2}{a}\right) + 3c\left(\frac{z-b}{a}\right) + d = 0$$

i.e, $\frac{(z-b)^3}{a^2} + 3b\frac{(z-b)^2}{a^2} + 3c\frac{(z-b)}{a} + d = 0.$

i.e, $(z-b)^3 + 3b(z-b)^2 + 3ac(z-b) + a^2d = 0$

i.e, $z^3 - 3z^2b + 3b^2z - b^3 + 3b(z^2 - 2bz + b^2) +$
$3acz - 3abc + a^2d = 0.$

i.e, $z^3 - 3z^2b + 3zb^2 - b^3 + 3z^2b - 6b^2z + 3b^2 + 3a$
$- 3abc + a^2d = 0.$

i.e, $z^3 - 3b^2z + 2b^3 + 3acz - 3abc + a^2d = 0.$

i.e, $z^3 + z(3ac - 3b^2) + (2b^3 - 3abc + a^2d) = 0.$

i.e, $z^3 + 3z(ac - b^2) + (2b^3 - 3abc + a^2d) = 0.$ (1)

i.e, $z^3 + 3Hz + G = 0$ (where $H = ac - b^2$ and
$G = 2b^3 - 3abc + a^2d$)

in order to solve the above cubic (which is known as Cardon's reducing cubic) Cardon took

$$z = p^{1/3} + q^{1/3}$$

$$z^3 = p + q + 3p^{1/3}q^{1/3}(p+q)$$

$$z^3 = p + q + 3p^{1/3}q^{1/3}z$$

i.e, $z^3 - 3p^{1/3}q^{1/3}z - (p+q) = 0$ —— (2)

Then it is clear that this cubic (5) identical with the cubic (1). Hence equating the coefficient in eqn (1) & (2) we have

$$H = -p^{1/3} q^{1/3}$$

$$\therefore p^{1/3} q^{1/3} = -H$$

$$G = -(p+q)$$

$$\therefore p+q = -G.$$

$$pq = -H^3$$

$$(p-q)^2 = (p+q)^2 - 4pq$$

$$= G^2 - 4(-H^3) = G^2 + 4H^3.$$

$$p-q = \sqrt{G^2+4H^3}$$

Thus we have, $p+q = -G$

$$p-q = \sqrt{G^2+4H^3}$$

$$\therefore 2p = -G + \sqrt{G^2+4H^3}$$

$$\therefore p = \frac{-G + \sqrt{G^2+4H^3}}{2}$$

and $2q = -G - \sqrt{G^2+4H^3}$

$$\therefore q = \frac{-G - \sqrt{G^2+4H^3}}{2}$$

Thus we have succeeded in finding out the separate values of p & q. it is clear that $p^{1/3}$ has three values p, wp, w^2p. Similarly $q^{1/3}$ has three values q, wq, w^2q. Thus the roots of reducing cubic will be

$$p^{1/3}+q^{1/3},\quad wp^{1/3}+wq^{1/3},\quad w^2p^{1/3}+w^2q^{1/3}$$