Dept. Of Maths.
B.Sc. Part I
(Maths. Hons)
Paper. I
(Theory of Equation) | Dt-17.01.22

Q:- State and prove Factor Theorem

Statement:- Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_{n-1}x + a_n = 0$ be an equation of n^{th} degree. Let be its root. Then we have to prove that $(x-\alpha)$ is a factor of $f(x)=0$

As α is a root of $f(x)=0$.

we have $f(\alpha)=0$ —— (1)

If $x-\alpha$ is to be a factor of $f(x)=0$. Then if we divide $f(x)$ by $x-\alpha$, then there should be no remainder. The division is not exact. Let $\phi(x)$ be the quotient and R be the remainder.

$$f(x) = \phi(x)(x-\alpha)+R$$

Put $x=\alpha$, $f(\alpha) = \phi(\alpha)(\alpha-\alpha)+R$

$$f(\alpha) = 0+R \text{ —— (from (1))}$$

This shows that the division of $f(x)$ by $(x-\alpha)$ is exact. Hence $(x-\alpha)$ is a factor.

Q. The imaginary roots of an equation with real coefficients always occur in conjugate pair.

Proof:- Let $f(x) = a_0 x^n + a_n x^{n-1} + \cdots$
$\cdots + a_{n-1} x + a_n = 0$ be a general Polynomial of nth degree.

Let $\alpha + i\beta$ be one of its roots, then we have to prove that $\alpha - i\beta$ must be another root of this equation.

As $\alpha + i\beta$ is given to be roots of the equation $f(x) = 0$. we have

$$f(\alpha + i\beta) = 0 \qquad \text{--- (1)}$$

Also from factor theorem it is clear that $f\{x - (\alpha + i\beta)\}$ must be factor of the equation. if $(\alpha - i\beta)$ be a root of the equation. then $x - (\alpha - i\beta)$ should be another factor of the equation.

i.e, $(x - \alpha - i\beta)(x - \alpha + i\beta)$ should be a factor

i.e, $\{(x-\alpha)^2 - i^2\beta^2\}$ should be a factor

i.e, $\{(x-\alpha)^2 + \beta^2\}$ should be a factor

The above expression will be a factor of the given equation. if we divide $f(x)$ by $\{(x-\alpha)^2 + \beta^2\}$ then there should be no remainder.

(3)

Let us, suppose that when actual division is done $\phi(x)$ is the quotient and $px+\theta$ is the remainder.

Thus we have $f(x) = \{(x-\alpha)^2+\beta^2\}\phi(x)+px+\theta$

Now, we put $x = \alpha+i\beta$ then

$$f(\alpha+i\beta) = \{(\alpha+i\beta-\alpha)^2+\beta^2\}\phi(\alpha+i\beta) + p(\alpha+i\beta)+\theta$$

$$\text{i.e, } 0 = \{i^2\beta^2+\beta^2\}\phi(\alpha+i\beta)+p(\alpha+i\beta)+\theta$$

$$\text{i.e, } 0 = (-\beta^2+\beta^2)\phi(\alpha+i\beta)+p(\alpha+i\beta)+\theta$$

$$\text{i.e, } p(\alpha+\beta)+\theta = 0.$$

$$p\alpha+\theta+ip\beta = 0.$$

But we know that if a complex numbe is equal to zero. Then its real and ima ginary part are separately equal to ze

$$p\alpha+\theta = 0, \quad p\beta = 0.$$

$$\text{i.e, } p = 0 \text{ as } \beta \neq 0$$

$$\therefore 0\alpha+\theta = 0 \Rightarrow \theta = 0.$$

As p and θ are separately equal to zero! it is obvious that the remaind $px+\theta = 0$, Thus we observe that the division of $f(x)$ is exactly by the pr ct of factors.

Hence $\alpha - i\beta$ is another proof.