Dept. of Mathematics
Class - B. Sc. Part II
(Maths. Subsidiary).

Name Of the Topic :- Leibnitz theorem
Paper - II.

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Leibnitz Theorem :

Statement :- if u and v are two functions of x which passes derivatives of n^{th} order, then

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n.$$

Proof :-

By method Of induction

Let $y = uv$
where u and v are functions of x.

By directly differentiating successively, we get

$$y_1 = u_1 v + u v_1$$

$$y_2 = (u_2 v + u_1 v_1) + (u v_2 + v_1 u_1)$$

$= u_2v + 2u_1v_1 + uv_2$

$= u_2v + 2C_1u_1v_1 + 2C_2uv_2$

$y_3 = (u_3v + u_2v_1) + 2(u_1v_2 + u_2v_1) + (uv_3 + u_1v_2)$

$= u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$

$= u_3v + 3C_1u_2v_1 + 3C_2u_1v_2 + 3C_2uv_3$

thus we see that this theorem is true for $n = 1, 2, 3$. According to the law of induction, we assume that this theorem is true for $n = m$ and we shall prove that this will also be true for $n = m+1$ and since this is true for particular value of $n = 1, 2, 3 ....$ therefore it will be true for every value of n.

Now, we assume that this theorem holds for $n = m$ i.e, we shall get the same formal expression for y_m which will be obtained by putting $n = m$ in the statement of the theorem. That is,

$$y_m = u_mv + {}^mC_1u_{m-1}v_1 + {}^mC_2u_{m-2}v_2 + ---- + {}^mC_{r-1}u_{m-r+1}v_{r-1} + {}^mC_ru_{m-r}v_r + ---- + {}^mC_{m-1}u_1v_{m-1} + {}^mC_muv_m \quad (1)$$

Differentiating once, we get

$$y_{m+1} = (u_{m+1}v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1}v_2)$$
$$+ {}^mC_2(u_{m-1}v_2 + u_{m-2}v_3) + \cdots$$
$$+ {}^mC_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}v_r)$$
$$+ {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \cdots$$
$$+ {}^mC_{m-1}(u_2 v_{m-1} + u_1 v_m) + {}^mC_m(u_1 v_m + u v_{m+1})$$

$$= u_{m+1}v + u_m v_1({}^mC_0 + {}^mC_1) + u_{m-1}v_2({}^mC_1 + {}^mC_2)$$
$$+ \cdots$$
$$+ u_{m-r+1}v_r({}^mC_{r-1} + {}^mC_r) + \cdots$$
$$+ u_1 v_m({}^mC_{m-1} + {}^mC_m) + {}^mC_m u v_{m+1}$$

Thus we see that if we assume the theorem to be true for a particular value of $n = m$, then this thm is also true for the next higher integer $n = m+1$.

But we have shown before that this theorem is true for $n = 2, 3$, therefore it is true for $n = 4$ and since this is true for $n = 4$. Hence this is true for $n = 5$.

Hence this theorem is true for every integral value of n.

Thus the theorem is proved