

Mathematics

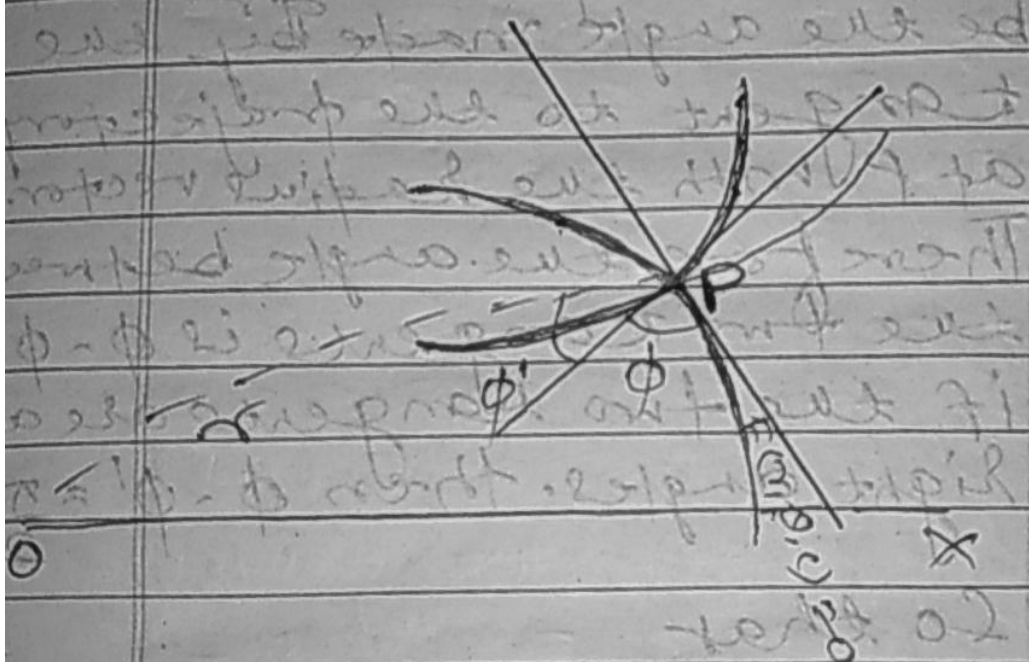
B.Sc. Part-II (Maths Hon)

Paper-IV

Name of the Topic:

Polar Co.ordinates
(Diff. eqn)

* To find the orthogonal trajectories of the family of curves $f(r, \theta, c) = 0$, c being a parameter.



The equation of the curve is given to be $f(r, \theta, c) = 0$.
Like before, differentiating $\textcircled{1}$ w.r.t. θ and eliminating c between $\textcircled{1}$ and the desired result, we shall obtain the

differential equation of the family. Let the differential equation be

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{--- (2)}$$

Let P be the point of intersection of any member of (1) with the trajectory. Let ϕ be the angle made by the tangent to (1) at P with the (common) radial vector. So that $\tan \phi = r \frac{d\theta}{dr}$. Let ϕ' be the angle made by the tangent to the trajectory at P with the radial vector.

Therefore the angle between the two tangents is $\phi - \phi'$. If the two tangents are at right angles, then $\phi - \phi' = \frac{\pi}{2}$.

So that

$$\tan(\phi - \phi') = \tan \frac{\pi}{2}$$

$$\Rightarrow \tan \phi - \tan \phi' = 0$$

$$\tan \phi = \tan \phi'$$

$$\Rightarrow 1 + \tan\phi \tan\phi' = 0$$

$$\Rightarrow \tan\phi \tan\phi' = -1$$

$$\Rightarrow \tan\phi' = \frac{-1}{\tan\phi}$$

$$= -\frac{1}{r} \frac{dr}{d\theta}$$

Thus if ϕ and ϕ' be the angles made by tangents to (1) and its trajectories and if the trajectories are orthogonal. Then we should have

$$\tan\phi' = -\cot\phi = -\frac{1}{r} \frac{dr}{d\theta}$$

It amounts to saying that the equation of the orthogonal trajectories will be obtained by replacing $r \frac{d\theta}{dr}$ (corresponding to $\tan\phi$) by $-\frac{1}{r} \frac{dr}{d\theta}$ i.e. replacing $\frac{dr}{d\theta}$ by $-\frac{r^2 d\theta}{dr}$.

Hence the differential equation of the family of orthogonal trajectories from (2) is

$$F\left(r, \theta, -\frac{r^2 d\theta}{dr}\right) = 0.$$

Q.1. Prove that the orthogonal trajectories of the rectangular hyperbola $xy = a^2$ is $x^2 - y^2 = c$.

Soln: Given $xy = a^2$ — (1)
Differentiating (1) w.r.t. x .
we get, $x \frac{dy}{dx} + y = 0$.

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \quad \text{--- (2)}$$

In order to get the orthogonal trajectories of (1), we have to replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2).

The differential equation of the orthogonal trajectory of the given curve is

$$-\frac{dx}{dy} = -\frac{y}{x} \Rightarrow \frac{dx}{dy} = \frac{x}{y}$$

$$\Rightarrow x dx = y dy$$

$$\Rightarrow \int x dx = \int y dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + K$$

$$\Rightarrow x^2 - y^2 = 2K = c^2 \quad (\text{by writing } 2K \text{ as } c^2)$$

Ans. Given proof is correct.
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