MATHS.
B.Sc. Part-II (Maths. Hons)
Paper-IV
Topic: Differential Equation (Orthogonal trajectories)

~~Q~~: Find the orthogonal trajectories of the family of coaxial circles

$$x^2+y^2+2gx+c=0$$ where g is the parameter.

Soln:- Differentiating the given equation, we get

$$2x+2y\frac{dy}{dx}+2g=0$$

$$\Rightarrow g=-\left(x+y\frac{dy}{dx}\right)$$

Putting the value of g in the given equation, thereby eliminating g, we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)+c=0$$

$$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}+c=0 \quad \text{---(1)}$$

Which is the differential equation to the given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (1). The differential eqn. of the orthogonal trajectories is

$$y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0 \quad \text{---(2)}$$

In order to solve (2), we put $x^2 = u$. So that on differentiating w.r.t. y, we get

$$2x\frac{dx}{dy} = \frac{du}{dy}$$

Therefore (2) becomes

$$y^2 - u + y\frac{du}{dy} + c = 0$$

$$\Rightarrow y\frac{du}{dy} - u = -y^2 - c$$

$$\Rightarrow \frac{du}{dy} - \frac{u}{y} = -y - \frac{c}{y} \quad \text{---(3)}$$

The above equation (3) is linear and its I.F. $= e^{\int -\frac{1}{y}dy}$

$$= e^{-\log y} = \frac{1}{y}$$

Hence the solution of (3) is

$$u \cdot \frac{1}{y} = \int \left(-y - \frac{c}{y}\right)\left(\frac{1}{y}\right) dy$$

$$= \int \left(-1 - \frac{c}{y^2}\right) dy$$

$$= -y + \frac{c}{y} + K$$

$$\Rightarrow \frac{x^2}{y} = -y + \frac{c}{y} + K$$

On putting $u = x^2$

$$\Rightarrow x^2 = -y^2 + c + Ky$$

$$\Rightarrow x^2 + y^2 - Ky - c = 0$$

which is of the form

$$x^2 + y^2 + 2fy - c = 0.$$

Q. Prove that the system of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Sol: Given $y^2 = 4a(x+a)$ — (1)

Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \quad \text{— (2)}$$

$$y^2 = 2xy\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$$= 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} - y^2 = 0$$

For orthogonal trajectory we shall have to replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (b)

Hence $y^2\left(-\frac{dx}{dy}\right)^2 + 2xy\left(-\frac{dx}{dy}\right) - y^2 = 0$

Eliminating (a) from (1) we get

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$$= 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} - y^2 = 0 \quad \text{— (b)}$$

For orthogonal trajectory, we shall have to replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (b).

Hence $y^2\left(-\frac{dx}{dy}\right)^2 + 2xy\left(-\frac{dx}{dy}\right) - y^2 = 0$

$\Rightarrow y^2\left(\frac{dx}{dy}\right)^2 - 2xy\left(\frac{dx}{dy}\right) - y^2 = 0$

$\Rightarrow y^2 - 2xy\left(\frac{dy}{dx}\right) - y^2\left(\frac{dy}{dx}\right)^2 = 0$

$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 + 2xy\left(\frac{dy}{dx}\right) - y^2 = 0$

multiplying by -1.
which is the same as (3)
Hence the given system is
self-orthogonal.

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