B. Sc. Part-II (Maths Hons)
Paper - IV

Topic: Orthogonal trajectory (diff. eqn).

Find the orthogonal trajectory of the family of circles $x^2+y^2=2ax$ each of which touches the y-axis at the origin.

Soln: Here, given $x^2+y^2=2ax$ — (1)

Differentiating (1) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}=2a$$

$$\Rightarrow x+y\frac{dy}{dx}=a \quad \text{— (2)}$$

Eliminating (a) between (1) and (2), we get,

$$x^2+y^2=2x\left(x+y\frac{dy}{dx}\right)$$

$$\Rightarrow x^2+y^2=2x^2+2xy\frac{dy}{dx}$$

$$\Rightarrow x^2+2xy\frac{dy}{dx}-y^2=0 \quad \text{— (3)}$$

The differential equation of the orthogonal trajectory is obtained by replacing $\frac{dy}{dx}$

by $\frac{-1}{\frac{dy}{dx}}$ and hence we get

$$x^2 + 2xy\left(\frac{-1}{\frac{dy}{dx}}\right) - y^2 = 0$$

$$\Rightarrow x^2 - 2xy\frac{dx}{dy} - y^2 = 0.$$

$$\Rightarrow y^2 - x^2 + 2xy\frac{dx}{dy} = 0 \quad \text{--- (4)}$$

We find that (3) and (4) are the same except that x and y are interchanged.

Hence the integral of (4) are the same except that x and y are interchanged and must be $y^2 + x^2 = 2by$ i.e. $x^2 + y^2 = 2by$ which is another set of circles, each of which touches the x-axis at the origin.

Qn. Find the orthogonal trajectory of $r^n \sin n\theta = a^n$?

Solⁿ: Given $r^n \sin n\theta = a^n$

taking logarithm on both side.

we get,

$$\log r^n + \log \sin n\theta = n \log a.$$

$$\Rightarrow n \log r + \log \sin n\theta = n \log a$$

Now differentiating w.r.t θ we get

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} + \frac{\cos n\theta}{\sin n\theta} \cdot n = 0.$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} + \cot n\theta = 0$$

$$\Rightarrow r \frac{d\theta}{dr} = -\tan \theta.$$

Therefore the differential equation of the orthogonal trajectory which is obtained by replacing $r\frac{d\theta}{dr}$ by $-\frac{1}{r} \cdot \frac{dr}{d\theta}$ is

$$-\frac{1}{r} \cdot \frac{dr}{d\theta} = -\tan \theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \tan \theta.$$

$$\Rightarrow \frac{dr}{r} = \tan n\theta \, d\theta$$

$$\Rightarrow \int \frac{dr}{r} = \int \tan n\theta \, d\theta$$

$$\Rightarrow \log r = \frac{1}{n} \log \sec n\theta + K.$$

$$\Rightarrow \log r = \frac{1}{n} \log \sec n\theta + \log c$$

$$\Rightarrow \log\frac{r}{c} = \frac{1}{n}\log\sec n\theta$$

$$\Rightarrow n\log\frac{r}{c} = \log\sec n\theta$$

$$\Rightarrow \log\left(\frac{r}{c}\right)^n = \log\sec n\theta$$

$$\Rightarrow \left(\frac{r}{c}\right)^n = \sec n\theta$$

$$\Rightarrow r^n = c^n \sec n\theta$$

$$\Rightarrow r^n \cos n\theta = c^n$$

which is the required equation of the trajectory.

Imp Question:- Prove that the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$ is self-orthogonal.

Solⁿ:- Differentiating the given equation, we get

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2x}{a^2+\lambda} + \frac{2yp}{b^2+\lambda} = 0 \quad \text{where } p = \frac{dy}{dx}.$$

$$\Rightarrow x(b^2+\lambda) + yp(a^2+\lambda) = 0.$$

$$\Rightarrow (b^2x + a^2yp) + \lambda(x+yp) = 0.$$

$$\Rightarrow \lambda = -\frac{b^2x+a^2yp}{x+yp}.$$

$$\therefore a^2+\lambda = a^2 - \frac{b^2x+a^2yp}{x+yp}$$

$$= \frac{(a^2-b^2)x}{x+yp}.$$

and $$b^2+\lambda = b^2 - \frac{b^2x+a^2yp}{x+yp}$$

$$= -\frac{(a^2-b^2)yp}{x+yp}.$$

Putting the values of $a^2+\lambda$ and $b^2+\lambda$ in the given eqn, we get

$$\frac{x^2(x+yp)}{(a^2-b^2)x} - \frac{y^2(x+yp)}{(a^2-b^2)yp} = 1$$

$$\Rightarrow \frac{x+yp}{a^2-b^2}\left(x - \frac{y}{p}\right) = 1.$$

$$\Rightarrow (x+yp)\left(x-\frac{y}{p}\right) = a^2-b^2 \quad \text{--- (1)}$$

In order to find its orthogonal trajectory, we have to replace p by $-\frac{1}{p}$ and we get,

$$\left(x-\frac{y}{p}\right)(x+yp) = a^2-b^2 \quad \text{--- (2)}$$

Now (1) and (2) are the same and hence the system of conics is self-orthogonal.

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