Mathematics.
B.Sc. Part-I (M.H)
Paper-I
Topic: Gregory's Series
(H. Trigo.)

Theorem

To prove that if $-1 \leq x \leq +1$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \text{ to } \infty$$

Sol. According to the Art, we have

$$\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}.$$

Putting $\alpha = 1$ and $\beta = x$ in this, we have

$$\log(1+ix) = \frac{1}{2}\log(1+x^2) + i\tan^{-1}x \quad \text{---(1)}$$

(Only taking the principle value)

Also, when $|x| \leq 1$

$$\log(1+ix) = ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} - \frac{(ix)^4}{4} + \cdots$$

$$ix - \frac{x^2}{2} - \frac{ix^3}{3} + \frac{x^4}{4} + \frac{ix^5}{5} \dots$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{4} - \dots\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) \quad \text{--- (2)}$$

Now equating the imaginary parts from (1) and (2), we get

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \dots$$

when $|x| \le 1$ --- (3)

Putting $x = \tan\theta$ so that $\theta = \tan^{-1}x$. The above series can be written as

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots \text{ to } \infty \quad \text{--- (4)}$$

where $|\theta| \le \frac{\pi}{4}$ i.e. $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

Convergency of the Gregory's Series

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots \text{ ad inf.}$$

Here,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^n \tan^{2n+1}\theta}{2n+1} \cdot \frac{2n-1}{(-1)^{n-1}\tan^{2n-1}\theta}\right|$$

$$= \lim_{n\to\infty} \frac{2n-1}{2n+1} \cdot \tan^2\theta$$

$$= \tan^2\theta.$$

Hence, the series is convergent for all values of θ for which $|\tan^2\theta| < 1$

i.e, $-1 < \tan\theta < +1$

i.e, $n\pi - \frac{\pi}{4} < \theta < n\pi + \frac{\pi}{4}$

for $\tan\theta = \pm 1$, also, the series on the R.H.S is a convergent series, for by putting $\tan\theta = \pm 1$ the series on the R.H.S is an alternating series which is convergent.

Hence the Gregory's series is convergent provided $|\tan\theta| \le 1$.

i.e, $n\pi - \pi/4 \le \theta \le n\pi + \pi/4$.

Note: While discussing convergency of the Gregory series we had pointed out that the Gregory series is convergent provided θ lies between the interval of the form $n\pi - \pi/4$ to $n\pi + \pi/4$. Hence it follows as a consequences, that the series would not be convergent in an interval whose extreme ends are not of the form $n\pi \pm \pi/4$. For example, the Gregory series is not convergent if θ lies between, say

$$n\pi + \frac{\pi}{4} \text{ and } n\pi + \frac{3\pi}{4}.$$

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