MATHEMATICS
B. Sc. Part-II
Paper- IV (M.H)
Topic: Singular Solution
(Differential Equation)

Definition:- Let $f(x, y, p) = 0$ be a given differential equation whose general solution is $\phi(x, y, c) = 0$.

Both p and c-discriminant contain the equation to the envelope which is called singular solution. it satisfies the given differential equation but it is not contained in the general solution and cannot be derived by giving to c a particular value in the general solution $\phi(x, y, c) = 0$

Note:- it has to be noted that in the case of Clairaut's form of differential equation both p and c discriminant are the same.

For example!
Consider the diff. equation

$$y = px + p^2 \quad \text{--- (1)}$$

it is of Clairaut's form and its general solution is

$$y = cx + c^2 \quad \text{--- (2)}$$

This is obtained by replacing p by c.

From (1), we have

$$f(x, y, p) = p^2 + px - y = 0$$

and from (2), we have

$$\phi(x, y, c) = c^2 + cx - y = 0$$

Obviously p-discriminant and c-discriminant are the same and in each case it is $x^2 + 4y = 0$

Hence the singular solution is $x^2 + 4y = 0$ which is contained in both p and c-discriminants.

Q. Solve and find the singular solution of

$$p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$$

Soln. The given equation can be written as

$$(p^2x^2 - 2pxy + y^2) = a^2p^2 + b^2$$

$$\Rightarrow (px-y)^2 = a^2p^2 + b^2$$

$$\Rightarrow (y-px)^2 = a^2p^2 + b^2$$

$$\Rightarrow y - px = \pm\sqrt{a^2p^2+b^2}$$

$$\Rightarrow y = px \pm \sqrt{a^2p^2+b^2}$$

Which is in Clairaut's form and hence its solution is

$$y = cx \pm \sqrt{a^2c^2+b^2}$$

$$\Rightarrow (y-cx)^2 = a^2c^2+b^2$$

$$\Rightarrow c^2(x^2-a^2) - 2cxy + y^2 - b^2 = 0$$

Clearly both p and c discriminants are

$$4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow x^2y^2 - (x^2-a^2)(y^2-b^2) = 0$$

$$\Rightarrow b^2x^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Which therefore is the required singular solution.

Ex-2

Examine the equation $4xp^2 = (3x-a)^2$ for singular solution.

Solⁿ: From the given equation, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{4x}$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{3x-a}{2\sqrt{x}}$$

$$= \pm\left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right)$$

$$\Rightarrow \int dy = \pm \int \left(\frac{3}{2}\sqrt{x} - \frac{a}{2\sqrt{x}}\right) dx$$

$$\Rightarrow y + C = \pm\left\{\frac{3}{2}\cdot\frac{2}{3}x^{2/3} - \frac{a}{2}\cdot 2\sqrt{x}\right\}$$

$$= \pm\left\{x^{2/3} - a\sqrt{x}\right\}$$

$$= \pm\sqrt{x}\,(x-a)$$

$$\Rightarrow (y+c)^2 = x(x-a)^2$$

Clearly c-discriminant

$$\equiv x(x-a)^2 = 0.$$

and p-discriminant $\equiv x(3x-a)^2 = 0$

Hence the singular solution is $x=0$ which is contained ~~and~~

in both c and p discriminants

Note:

(i) $x-a=0$ which occurs twice in C-discriminant and is not included in p-discriminant is the nodal locus.

(ii) $3x-a=0$ which occurs twice in p-discriminant and is not included in C-discriminant represents tac locus.

Dr. Birendra Prasad
Asst. Prof.
Dept. of Maths
D.K. College,
Dumraon