

Mathematics.

B. Sc. Part-I (M.H)

Paper-I

Topic: Gregory's Series
(Higher Trigonometry)

Examples of Particular

(i) if θ lies between $\pi - \frac{\pi}{4}$ and

$\pi + \frac{\pi}{4}$ i.e. if θ lies between

$\frac{3\pi}{4}$ and $\frac{5\pi}{4}$ then we have, $n=1$

and from the eqn $\theta - \pi =$

$$t \tan \theta - \frac{1}{3} t \tan^3 \theta + \frac{1}{5} t \tan^5 \theta,$$

we get

$$\theta - \pi = t \tan \theta - \frac{1}{3} t \tan^3 \theta + \frac{1}{5} t \tan^5 \theta -$$

(ii) if $\pi - \frac{\pi}{4} < \theta < \pi$ i.e.

if $-\frac{5\pi}{4} \leq \theta \leq -\frac{3\pi}{4}$ then we have

$n=-1$ and we get from equation

$$\theta + \pi = t \tan \theta - \frac{1}{3} t \tan^3 \theta +$$

$$\frac{1}{5} t \tan^5 \theta - \dots - t \tan^{\infty} \theta$$

Evaluation of π

One of the chief uses of Gregory's Series is to find the value of π . Thus if we put $x=1$ in (3), then since $\tan^{-1} 1 = \frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

But the difficulty in using this series lies in the fact that we have to take a large number of terms in finding the value of π correct to a given number of decimal places. Therefore we take recourse to another series; e.g. $\pi = 2 \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \right]$

① Euler's Series: It can be proved easily that

$$\frac{\pi^2}{6} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

which when expanded by Gregory Series, becomes

$$= \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots \right]^2$$

$$+ \left[\frac{1}{3} - \frac{1}{3 \cdot 3} + \frac{1}{3 \cdot 3^3} - \frac{1}{3 \cdot 3^5} + \dots \right]$$

etc.

(ii) Machin's Series!

$$\therefore 2 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{2}{5}$$

$$= \tan^{-1} \frac{10}{24} = \tan^{-1} \frac{5}{12}$$

$$\therefore 4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \frac{10}{12} = \tan^{-1} \frac{120}{119}$$

$$1 - \frac{25}{144}$$

$$\therefore 4 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{1}{239}$$

$$= \tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239}$$

$$\therefore \pi = 4 \left[\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots \right]$$

$$\left[\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \frac{1}{7 \cdot 239^7} + \dots \right]$$

Now

$$\frac{1}{5} = .2$$

$$\frac{1}{3 \cdot 5^3 \cdot 375} = 0.0026666$$

$$\frac{1}{5 \cdot 5^5} = \frac{1}{15625} = 0.000064$$

$$\text{Also } \frac{1}{239} = 0.004184$$

$$\text{Hence } \frac{\pi}{4} = 4 \left[0.2 - 0.0026666 + 0.000064 - \dots \right]$$

$$= 0.7853983$$

$$\therefore \pi = 4 \times 0.7853983$$

$$= 3.14159$$

This is the correct value of π to 5 decimal places.

(iii) Rutherford's Series:

It can be shown that

$$\frac{1}{239} = \frac{1}{70} - \frac{1}{99}$$

Thus $\frac{\pi}{4} = \tan^{-1} \frac{1}{5}$

$\tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}$

Both the series of $\tan^{-1} \frac{1}{70}$ and $\tan^{-1} \frac{1}{99}$ are convenient for numerical calculation.

~~Ex 2.~~ Write down the value of n when θ lies between $\frac{19\pi}{4}$ and $\frac{21\pi}{4}$ for the truth of general theorem.

$\theta - n\pi = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta$

Soln We know that the general theorem, $\theta - n\pi = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta$

is valid if θ lies between $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$. It is given here that θ lies between

$\frac{19\pi}{4}$ and $\frac{21\pi}{4}$ i.e. $5\pi - \frac{\pi}{4}$ and $5\pi + \frac{\pi}{4}$

$n = 5$

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