Mathematics Hons.
B.Sc. Part - I
Paper - I
Name of the Topic:
Gregory's Series
(Higher Trigonometry)

Ex-1: Prove that

$$\tan^{-1}\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$=n\pi+\frac{\pi}{4}+\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\cdots$$

Soln:-

$$L.H.S=\tan^{-1}\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$=\tan^{-1}\frac{1+\tan\theta}{1-\tan\theta}$$

dividing the num. and den. by $\cos\theta$.

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$=n\pi+\frac{\pi}{4}+\theta$$

$$=n\pi+\frac{\pi}{4}+\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta$$ (by Gregory's series)

Ex-2 If $0 \le x \le 1$, Show that

$$\tan^{-1}\frac{1+x}{1-x} = \frac{\pi}{4} + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Soln:- Let $x = \tan\theta$.

Then since $0 \le x \le 1$, $0 \le \theta \le \frac{\pi}{4}$

Now, L.H.S $= \tan^{-1}\frac{1+x}{1-x}$

$$= \tan^{-1}\frac{1+\tan\theta}{1-\tan\theta}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4} + \tan^{-1}x$$

$$= \frac{\pi}{4} + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

(by Gregory's Series)

Ex If $-(\sqrt{2}-1) < x < \sqrt{2}-1$, Prove that $2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \text{to } \infty\right)$

$$= \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \cdots$$

Solution

$$L.H.S = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \text{ to } \infty\right)$$

$$= 2\tan^{-1}x \quad \text{Provided } |x| < 1$$

$$= \tan^{-1}\frac{2x}{1-x^2}$$

Now, $\tan^{-1}\frac{2x}{1-x^2}$ can be expanded by Gregory's series if $\left|\frac{2x}{1-x^2}\right| < 1$

i.e if $\tan^{-1}\frac{2x}{1-x^2} < 45^\circ$

i.e if $2\tan^{-1}x < 45^\circ$

i.e if $\tan^{-1}x < 22\frac{1}{2}^\circ$

But we know that

$$\tan 22\tfrac{1}{2}^\circ = \sqrt{2} - 1$$

Hence if $|x| < \sqrt{2} - 1$. (which is given here), then the Inverse function $\tan^{-1}\frac{2x}{1-x^2}$ can be expanded by Gregory's series.

Thus

$$\tan^{-1}\frac{2x}{1-x^2} = \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \ldots = R.H.S.$$

If $0 < \theta < \frac{\pi}{4}$, Prove that

$$\log(\sec\theta) = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \cdots$$

Sol^n:- Putting $\tan\theta = x$ in the R.H.S, we get

$$R.H.S = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots$$

But $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots ;\ |x| < 1.$

Now replacing x by (ix), we get

$$\log(1+ix) = ix - \frac{i^2x^2}{2} + \frac{i^3x^3}{3} - \frac{i^4x^4}{4} + \frac{i^5x^5}{5} - \frac{i^6x^6}{6} + \cdots$$

$$= ix + \frac{x^2}{2} - i\frac{x^3}{3} - \frac{x^4}{4} + i\frac{x^5}{5} + \frac{x^6}{6} - \cdots$$

$$= \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) \quad \text{—(1)}$$

Now, $\log(1+ix)$

$$= \frac{1}{2}\log(1^2+x^2) + i\tan^{-1}x$$

$$= \frac{1}{2}\log(1+x^2) + i\tan^{-1}x \quad \text{(2)}$$

Hence equating the real part from (1) and (2), we get

$$\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \cdots = \frac{1}{2}\log(1+x^2)$$

i.e., $$\frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \cdots = \frac{1}{2}\log(1+\tan^2\theta),$$

provided $0 < \tan\theta < 1$

i.e., $0 < \theta < \frac{\pi}{4}$

$$= \frac{1}{2}\log(\sec^2\theta) = \frac{1}{2}\cdot 2\log(\sec\theta)$$

$$= \log(\sec\theta) = \text{L.H.S.}$$

Dr. Birendra Prasad,
Asst. Prof. of Maths,
Dept. of Maths, D.K. College,
Dumraon