Mathematics Hons.

B. Sc. Part - I

Paper - I

Topic: Gregory's Series (Higher Mathematics)

Ex. Express $\tan^{-1}(\cos\theta + i\sin\theta)$ in the form $A + iB$ and deduce that

(i) $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \cdots\cdots = \frac{\pi}{4}$,

(ii) $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \cdots\cdots = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Sol:

We shall separate $\tan^{-1}(\cos\theta + i\sin\theta)$ in to real and imaginary parts in two ways.

Let $\tan^{-1}(\cos\theta + i\sin\theta) = A + iB$

so that $\tan(A + iB) = \cos\theta + i\sin\theta$

Taking its conjugate, we get

$\tan(A - iB) = \cos\theta - i\sin\theta$.

Now,

$\tan 2A = \tan\{(A + iB) + (A - iB)\}$

$$= \frac{\tan(A+iB) + \tan(A-iB)}{1 - \tan(A+iB)\tan(A-iB)}$$

$$= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{1 - (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2\cos\theta}{1 - (\cos^2\theta + \sin^2\theta)}$$

$$= \frac{2\cos\theta}{1-1}$$

$$= \frac{2\cos\theta}{0} = \infty$$

$$\therefore 2A = \frac{\pi}{2} \Rightarrow A = \frac{\pi}{4} \quad \text{--- (1)}$$

Again,

$$\tan(2iB) = \tan\{(A+iB) - (A-iB)\}$$

$$= \frac{\tan(A+iB) - \tan(A-iB)}{1 + \tan(A+iB)\cdot\tan(A-iB)}$$

$$= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{1 + (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2i\sin\theta}{2} = i\sin\theta$$

$$\Rightarrow i\tanh 2B = i\sin\theta$$

$$\Rightarrow \tanh 2B = \sin\theta$$

$\Rightarrow 2B = \tanh^{-1}(\sin\theta)$

$$= \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}$$

$$= \frac{1}{2}\log\frac{(\cos\theta/2+\sin\theta/2)^2}{(\cos\theta/2-\sin\theta/2)^2}$$

$$= \log\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}$$

$$= \log\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$= \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore B = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \quad (2)$$

Also from Gregory's Series,

$\tan^{-1}(\cos\theta + i\sin\theta)$

$$= (\cos\theta + i\sin\theta) - \frac{1}{3}(\cos\theta + i\sin\theta)^3$$

$$+ \frac{1}{5}(\cos\theta + i\sin\theta)^5 - \cdots$$

$= (\cos\theta + i\sin\theta) - \frac{1}{3}(\cos 3\theta + i\sin 3\theta) + \frac{1}{5}(\cos 5\theta + i\sin 5\theta) - \dots$

$= \left[\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta \dots\right]$

$+ i\left[\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \dots\right]$ — (3)

Hence equating the real and imaginary parts from (1), (2) and (3), we get

$\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \dots = \pi/4$

and $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \dots = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Hence the result.

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Q. Prove that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{2\sqrt{2}}$

Soln:- From the expansion of $\tan^{-1}(\cos\theta + i\sin\theta)$ we have

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$\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta -$
$\frac{1}{7}\cos 7\theta + \cdots = \pi/4.$

Putting $\theta = \pi/4$, we get

$$\cos\frac{\pi}{4} - \frac{1}{3}\cos\frac{3\pi}{4} + \frac{1}{5}\cos\frac{5\pi}{4} - \frac{1}{7}\cos\frac{7\pi}{4} + \cdots = \frac{\pi}{4}.$$

$$\Rightarrow \frac{1}{\sqrt{2}} - \frac{1}{3}\cos\left(\pi - \frac{\pi}{4}\right) + \frac{1}{5}\cos\left(\pi + \frac{\pi}{4}\right) - \frac{1}{7}\cos\left(2\pi - \frac{\pi}{4}\right) + \cdots = \frac{\pi}{4}.$$

$$\Rightarrow \frac{1}{\sqrt{2}} + \frac{1}{3}\cos\frac{\pi}{4} - \frac{1}{5}\cos\frac{\pi}{4} - \frac{1}{7}\cos\frac{\pi}{4} + \cdots = \pi/4.$$

$$\Rightarrow \frac{1}{\sqrt{2}} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}} - \frac{1}{5}\cdot\frac{1}{\sqrt{2}} - \frac{1}{7}\cdot\frac{1}{\sqrt{2}} + \cdots = \pi/4.$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right] = \pi/4$$

$$\text{i.e. } 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \times \sqrt{2}$$

$$= \frac{\pi}{2\sqrt{2}}$$

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