Mathematics Hons.
B. Sc. Part - II
Paper - IV
Topic: Linear Equations with Constant coefficients.
(Differential Equation)

Ex. (1) Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Sol^n: The auxiliary equation is $m^2 - 5m + 6 = 0$.

$\Rightarrow (m-2)(m-3) = 0$

$\therefore m = 2$ or 3.

$\therefore$ The general solution is

$y = C_1 e^{2x} + C_2 e^{3x}$.

(2) Solve $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

Sol^n: The auxiliary eq^n is

$m^2 - m - 2 = 0$.

$\Rightarrow m^2 - 2m + m - 2 = 0$

$\Rightarrow (m+1)(m-2) = 0$

$\therefore m = -1$ or 2.

$\therefore$ the general solution is

$y = C_1 e^{-x} + C_2 e^{2x}$.

(3) Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$

Solⁿ:- The auxiliary eqⁿ is

$m^3 - 6m^2 + 11m - 6 = 0.$

$\Rightarrow (m-1)(m-2)(m-3) = 0.$

$\therefore m = 1, 2$ or $3.$

The general solution is

$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$

(4) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0,$

where $x = 2$, $\frac{dx}{dt} = 0$ when $t = 0$

Solⁿ: The auxiliary eqⁿ is

$m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\therefore m = 1$ or 2

$\therefore$ The general solution is

$x = c_1 e^{t} + c_2 e^{2t}$ —— (1)

Now, we evaluate c_1 and c_2 with the given conditions

when $t = 0$, we have $x = 2$,

Hence from (1)

$$2 = c_1 + c_2 \quad \text{——— (2)}$$

Now, differentiating (1) w.r.t. t, we get

$$\frac{dx}{dt} = c_1 e^t + c_2 2e^{2t}$$

Putting $t=0$, $\frac{dx}{dt} = 0$,

we get $0 = c_1 + 2c_2$

Solving (2) and (3), we get

$c_2 = -2$ and $c_1 = 4$.

Hence, putting $c_1 = 4$ and $c_2 = -2$ in (1), the required solution is $x = 4e^t - 2e^{2t}$.

CASE II: Roots real and equal

Let m_1 be a root repeated twice (for example) of the auxiliary equation $f(m) = 0$ then,

$f(m) = \psi(m)(m - m_1)^2$ which

$\Rightarrow f(D) = \psi(D)(D - m_1)^2$.

Hence any value of y that makes $(D - m_1)^2 y = 0$ must also make

$f(D)y = \psi(D)(D - m_1)^2 y = 0$.

Therefore solution of $(D-m_1)^2 y = 0$ is also a solution of $f(D)y = 0$.

We now proceed to find the solution of $(D-m_1)^2 y = 0$. The above equation can be written as $(D-m_1)(D-m_1)y = 0$ — (1)

Let $(D-m_1)y = Z$ — (2)

So, that from (1), we have $(D-m_1)Z = 0$

from case (1), its solution is

$$Z = C_1 e^{m_1 x}$$

Substituting this value of Z in (2), we get

$$(D-m_1)y = C_1 e^{m_1 x}$$

$$\Rightarrow \frac{dy}{dx} - m_1 y = C_1 e^{m_1 x} \quad \text{— (3)}$$

This is a linear equation of the first order.

Here,

$$I.F = e^{\int P dx} = e^{\int -m_1 dx} = e^{-m_1 x}$$

Hence, the solution is

$$y e^{-m_1 x} = C_1 \int e^{m_1 x} \cdot e^{-m_1 x} dx + C_2$$

$= c_1 \int dx + c_2$

$= c_1 x + c_2$

$y = (c_1 x + c_2) e^{m_1 x}$

Thus $(c_1 + c_2 x) e^{m_1 x}$ is a part of the complimentary function.

Similarly, it can be shown that if a root m_1 is repeated three times, then $(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$ will form part of the C.F. and so on.

Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Solⁿ: The auxiliary eqⁿ is

$m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0, \therefore m = 3, 3.$

That is, the root $m = 3$ is repeated twice.

Hence the general solution is

$y = (c_1 + c_2 x) e^{3x}$.

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