Mathematics Hons.

B.Sc. Part - II

Paper - IV

Topic: Linear equations with constant coefficients (diff. eqn.)

CASE III :- Roots conjugate complex and not repeated.

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ be the roots of the auxiliary equation. Then the general solution of (1) is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$y = c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$$

$$= e^{\alpha x}\{c_1 e^{i\beta x} + c_2 e^{-i\beta x}\}$$

$$= e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\}$$

$$= e^{\alpha x}\{(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x\}$$

$$= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ are arbitrary constants.

Note: Let it be noted that α is the real part and β is the imaginary part of the complex roots $\alpha \pm i\beta$.

CASE IV :- Roots conjugate complex and repeated.

When the conjugate complex roots $\alpha \pm i\beta$ are each repeated r times, the corresponding part of the value of the complementary function is

$$e^{\alpha x}\{(A_1 + A_2 x + \dots + A_r x^{r-1}) \cos\beta x + (B_1 + B_2 x + \dots + B_r x^{r-1}) \sin\beta x\}$$

it is an extension of the case No. III.

Ex. Solve $\frac{d^2y}{dx^2} + 4y = 0$

Soln:- The auxiliary eqn is

$$m^2 + 4 = 0$$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m = \pm 2i$$

(Here real part is = 0 and imaginary part = 2)

$\therefore$ The general solution is

$$y = e^{0 \cdot x}(c_1 \cos 2x + c_2 \sin 2x)$$

i.e, $y = c_1 \cos 2x + c_2 \sin 2x.$

Ex. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Soln:- The auxiliary eqn is

$$m^2 + m + 1 = 0.$$

Solving the equation, we get

$$m = \frac{-1 \pm \sqrt{1-4 \cdot 1 \cdot 1}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$

Here real part $= \frac{-1}{2}$ and

imaginary part $= \frac{\sqrt{3}}{2}$

$\therefore$ The general solution is

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$$

Ex. Solve $\frac{d^3y}{dx^3} - 8y = 0.$

Sol^n: The auxiliary eq^n is

$m^3 - 8 = 0$

$\Rightarrow (m-2)(m^2+2m+4) = 0$

$\therefore m = 2$ or $m^2+2m+4=0$

Solving $m^2+2m+4=0$, we get

$m = \frac{-2 \pm \sqrt{4-4.1.4}}{2}$

$= \frac{-2 \pm 2\sqrt{3}i}{2}$

$= -1 \pm \sqrt{3}i$

Hence $m = 2, -1 \pm \sqrt{3}i$

(real part $= -1$ and imaginary part $= \sqrt{3}$)

$\therefore$ The general solution is

$y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$.

(b) Solve $\frac{d^4y}{dx^4} - a^4y = 0$

Sol^n: The auxiliary equation is

$$m^4 - a^4 = 0.$$

$$\Rightarrow (m^2 - a^2)(m^2 + a^2) = 0.$$

from $m^2 - a^2 = 0$

$$m^2 = a^2$$

and

$$m^2 + a^2 = 0$$

$$\therefore m^2 = -a^2$$

$$\Rightarrow m = \pm a, \pm ia.$$

Therefore the general solution is

$$y = C_1 e^{ax} + C_2 e^{-ax} + (C_3 \cos ax + C_4 \sin ax).$$

~~Prasad~~

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